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Simulation of simply cross correlated random fields by series expansion methods

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Abstract

A practical framework for generating cross correlated random fields with a specified marginal distribution function, an autocorrelation function and cross correlation coefficients is presented in the paper. The approach relies on well-known series expansion methods for simulation of a Gaussian random field. The proposed method requires all cross correlated fields over the domain to share an identical autocorrelation function and the cross correlation structure between each pair of simulated fields to be simply defined by a cross correlation coefficient. Such relations result in specific properties of eigenvectors of covariance matrices of discretized field over the domain. These properties are used to decompose the eigenproblem, which must normally be solved in computing the series expansion, into two smaller eigenproblems. Such a decomposition represents a significant reduction of computational effort.

Non-Gaussian components of a multivariate random field are proposed to be simulated via memoryless transformation of underlying Gaussian random fields for which the Nataf model is employed to modify the correlation structure. In this method, the autocorrelation structure of each field is fulfilled exactly while the cross correlation is only approximated. The associated errors can be computed before performing simulations and it is shown that the errors happen only in the cross correlation between distant points and that they are negligibly small in practical situations.

Some comments on available techniques for simulation of underlying random variables in connection with the accuracy of basic fields' statistics at a given sample size are made. For this purpose a simple error assessment procedure is presented.

Simulated random fields can be used both for representation of spatially correlated properties of structure or random load in the stochastic finite element method (SFEM). An example of this application is related to size effect studies in the nonlinear fracture mechanics of concrete, and is used to illustrate the method.

Keywords: Multivariate random field; Orthogonal transformation of a covariance matrix; Karhunen–Loève expansion; SFEM; Monte Carlo simulation; Latin hypercube sampling; Size effect

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Nomenclature

$\mathbf{C}$	cross correlation matrix of random fields
$\mathbf{D}$	block correlation matrix with cross correlation on the diagonals
$\mathbf{F_{uu}}$	autocorrelation matrix over N discretization points
$\mathbf{F}$, $\mathbf{F}'$	full-block correlation matrix, its approximation
$c_{N_{\text{var}}}$	correction (normalization) factor for eigenvalues measure of captured variability
$C_{HH}()$	autocovariance/correlation function
d_i	correlation lengths
E[]	expectation (the mean value)
Corr[]	correlation
Var[]	variance
$H(\boldsymbol{x})$, $\tilde{H}(\boldsymbol{x})$	original and approximated random field
$H(\boldsymbol{x},\theta_0)$	*realization* of the field
$\mathbf{I}$	identity matrix
$\psi_j(\boldsymbol{x})$	orthogonal eigenfunctions of a covariance kernel
N	number of discretization points
N_{sim}	number of simulations, realizations (sample size)
N_{var}	number of random variables number of eigenmodes of autocorrelation function/matrix
N_F	number of random fields number of eigenmodes of cross correlation matrix
$N_{F,r}$	reduced number of eigenmodes of cross correlation matrix
N_r	number of random variables needed for N_F random fields number of considered eigenmodes of matrix $\mathbf{F}$
N_{run}	number of runs of the set of N_{sim} simulations
$\mathbf{S}(\theta_0)$	correlation matrix of a realization of random vector
$\{h_j(\boldsymbol{x})\}_{j=1}^{\infty}$	orthogonal functions
$\mathbf{u}=\{H(\boldsymbol{x}_1),\ldots,H(\boldsymbol{x}_N)\}$	a vector of nodal field values
$\mathbf{u}_j=\{H_j(\boldsymbol{x}_1),\ldots,H_j(\boldsymbol{x}_N)\}$	a vector of nodal values of the field j
$\Sigma_{H(\boldsymbol{x})\mathbf{u}}$	covariance matrix between $\mathbf{u}$ and the field values at $\boldsymbol{x}\in\Omega$
$\Sigma_{\mathbf{uu}}$	covariance matrix of the nodal points
$\boldsymbol{\mu}_u$	mean value of the field over its nodal values $\mathbf{u}$
$\mathbf{\Phi^C}$, $\mathbf{\Phi^D}$, $\mathbf{\Phi^F}$, $\mathbf{\Phi^u}$	orthonormal eigenvectors of correlation matrices $\mathbf{C}$, $\mathbf{D}$, $\mathbf{F}$, $\mathbf{F_{uu}}$
$\mathbf{\Lambda^C}$, $\mathbf{\Lambda^D}$, $\mathbf{\Lambda^F}$, $\mathbf{\Lambda^u}$	diagonal eigenvalue matrix of correlation matrices $\mathbf{C}$, $\mathbf{D}$, $\mathbf{F}$, $\mathbf{F_{uu}}$
θ	random nature
θ_0	a given outcome
$\mu(\boldsymbol{x})$	mean function of expanded random field
$\lambda_j^{\mathbf{u}}$	eigenvalues of the covariance kernel
σ^2	variance of the field, variable
$\boldsymbol{\chi}^{\mathbf{D}}$	block sample matrix (Gaussian)
$\boldsymbol{\xi}$	vector of independent standard Gaussian random variables
ε_C	error measure introduced by the reduction $N_F \to N_{F,r}$
ε_S	matrix norm (error) of $\mathbf{S}$
$\boldsymbol{\varepsilon}_{\mathbf{F}}$	symmetric correlation error matrix

1. Introduction

The random nature of many features of physical events is widely recognized both in industry and by researchers. The randomness of a gust wind, random structural features of materials, random fluctuations in temperature, humidity, and other environmental factors, all make the characterization provided by deterministic models of mechanics less satisfactory with respect to predictive capabilities. However, the entire prob-

lem of uncertainty and reliability can be addressed in a mathematically precise way and the random characteristics of nature can be addressed by computational models. For example, spatially fluctuating values of material parameters can be conveniently represented by means of random fields (e.g. strength, modulus of elasticity, fracture energy, etc.). Except for the narrow class of problems that can be solved analytically, the solution to the variety of complex engineering problems involving uncertainty regarding mechanical properties and/or the excitations they are subjected to must be found by means of simulation. The only currently available universal method for accurate solution of such stochastic mechanics problems is the Monte Carlo technique. Additionally, sensitivity and reliability analyses can be performed with minimal effort. The fast development of computer hardware enables scientists and engineers to carry out simulation-based analyses, partially also thanks to parallel implementation of problems on multiple computers.

One of the most important stages of the Monte Carlo type simulation technique as applied to systems with random properties (or systems subjected to random excitations) is the generation of sample realizations of these random properties. The generated sample functions must accurately describe the probabilistic characteristics of the corresponding stochastic processes or fields. Moreover, since the analyzed problems are usually computationally intensive (e.g. large scale nonlinear finite element computations), an analyst must select a simulation technique giving stable solutions with a small number of samples.

Although one of the most popular methods for simulation of random fields is the *spectral representation method* (SR) e.g. [46,48,47], the present paper is focused on methods involving orthogonal series expansion of correlation functions/matrices. This method is strongly related to the Karhunen–Loéve expansion (KLE) method. In the author's opinion, there are only a few papers focused on simulation techniques based on modal decomposition of autocorrelation functions. Of course, each autocorrelation function is associated with its spectral density function through the Wiener–Khintchine relationship. The equivalence between the SR and KLE for weakly stationary processes has recently been pointed by Grigoriu [21] who compared the Karhunen–Loéve, spectral and sampling representations for simulating various types of stochastic processes, including non-Gaussian processes.

Simulation of non-Gaussian processes is mostly based on memoryless transforms of the standard Gaussian processes. These processes are known as translation processes [17–19]. The central problem is to determine the corresponding Gaussian covariance matrix (or equivalently, the Gaussian power spectral density function) that yields the target non-Gaussian covariance matrix after the memoryless transformation. Yamazaki and Shinozuka [61] proposed an iterative algorithm for generating samples of non-Gaussian random fields with prescribed spectral density and prescribed marginal distribution function based on iterative updating of the power spectral density. Their algorithm was shown to have certain difficulties matching the marginal probability distributions in the case when the distributions were highly skewed and Deodatis and Micaletti [11] presented two modifications to it. Grigoriu [20] presented another model based on the spectral representation theorem for weakly stationary processes, that can match the second-moment properties and several higher order moments of any non-Gaussian process, and consists of a superposition of harmonics with uncorrelated but dependent random amplitudes. Recently, Sakamoto and Ghanem [44] and Puig et al. [42] utilized Hermite polynomial chaos method. In their method, the non-Gaussian processes are simulated by expanding the non-Gaussian distribution using Hermite polynomials with the standard Gaussian variable as argument. The correlation structure is decomposed according to the KLE of the underlying Gaussian process. The accuracy of this representation was studied by Field and Grigoriu [15] who pointed out some limitations of the approach. Grigoriu [21] criticize the algorithm for its computational intensity and questionable accuracy. For simulation of non-Gaussian weakly stationary stochastic processes with band-limited spectra, the sampling representation is recommended instead. Phoon et al. [40] recently proposed simulation of non-Gaussian processes via Karhunen–Loéve expansion with uncorrelated non-Gaussian coefficients of zero mean and unit variance. The key feature of their technique is that the distribution of the random coefficients (random variables) is updated iteratively. To achieve uncorrelatedness of the these random variables a slight modification of former correlation control algorithm was proposed so that the target distributions and covariance functions are matched.

In the present paper (which is an extended and translated version of the paper by Vořechovský [52]) two well-known methods are chosen for representation of a Gaussian random field, and based on these methods a simple extension to the simulation of the target type of multivariate stochastic fields is shown. The two methods represent a stochastic field in terms of a linear combination of deterministic functions/vectors $\varphi_j(\boldsymbol{x})$, and a

finite set of uncorrelated standard Gaussian random variables $\chi_j(\theta)$, where θ stands for the random nature. Both methods enable continuous representation of a field. After a brief review of these two methods in the context of univariate random fields (Section 2) we proceed to cross correlated Gaussian vector random fields (Section 3) and the proposed method. Section 4 shows how the presented approach can be extended for simulation of non-Gaussian vector random fields via transformations of an underlying Gaussian random field. In Section 5 a numerical example is presented to document the computational savings and small degree of error arising when simulating non-Gaussian fields. Section 6 proposes a simple error assessment procedure. Several alternatives of the Monte Carlo simulation methods at the level of random variable simulation are compared with respect to the accuracy of the simulated field's samples. Possible utilization of simulated fields is presented in Section 7, namely the modeling of material parameters aiming at capturing the statistical part of size effect on the strength of quasibrittle structures is sketched in words.

2. Series expansion methods for the simulation of a random field

In most applications, it is necessary to represent a continuous-parameter random field in terms of a vector of random variables. For example, the discrete nature of the finite element formulation also requires the random fields to be discretized – this process is known as discretization of a random field. For the sake of completeness, a brief list of several methods for discretization of random fields proposed in the past can be mentioned. These include the *point discretization methods*:

- the midpoint method e.g. [13],
- the shape function method e.g. [28,29],
- the integration point method e.g. [31,4],
- the optimal linear estimation method (OLE) [26],

and a group of *average discretization methods*, e.g.:

- the spatial average method (SA) e.g. [51,13,31],
- the weighted integral method e.g. [8,9,12].

The methods always involve a finite number of random variables having a straightforward interpretation: point values or local averages of the original random field $H(\boldsymbol{x},\theta)$. For a given outcome θ_0, $H(\boldsymbol{x},\theta_0)$ is a realization of the field. On the other hand, for a given $\boldsymbol{x}_0$ from Ω, $H(\boldsymbol{x}_0,\theta)$ is a random variable. The following Eq. (1) can be viewed as the *expansion* of each realization of the approximated field $\hat{H}(\boldsymbol{x},\theta_0) \in \mathscr{L}^2(\Omega)$ over the basis of all functions $\varphi_j(\boldsymbol{x})$ $(j = 1 \ldots N_{\text{var}})$, $\chi_j(\theta_0)$ being the coordinates. The discretization methods aim at expanding any realization of the original random field $H(\boldsymbol{x},\theta_0) \in \mathscr{L}^2(\Omega)$ over a complete set of deterministic functions, where $\boldsymbol{x} \in \Omega$ is a continuous parameter, and Ω is an open set of $\mathbb{R}^{\dim}$ describing the system geometry. By means of these random variables $\chi_j(\theta)$, the approximated random field $\tilde{H}(\boldsymbol{x})$ can be expressed as a finite summation (series expansion):

$$\hat{H}(\boldsymbol{x},\theta) = \sum_{j=1}^{N_{\text{var}}} \chi_j(\theta)\varphi_j(\boldsymbol{x}) \tag{1}$$

where the deterministic functions $\varphi_j(\boldsymbol{x})$ for different discretization methods are well-arranged in the recent report by Sudret and Der Kiureghian [49].

Autocovariance structure of a random field: As the randomness of the spatial variability in multidimensional nature is generally anisotropic, the autocorrelation function of the spatial variability is supposed to be a function of a non-Euclidean norm of two points $\boldsymbol{x},y \in \Omega$:$\|\boldsymbol{x},y\| = \{\|x_1,y_1\|,\ldots,\|x_{\dim},y_{\dim}\|\}$. If the covariance function depends on distance alone, the function is said to be isotropic. The following commonly used exponential form of an *autocorrelation function* is used as an example (for unit variance σ^2 of the field):

$$C_{HH}(\boldsymbol{x},\boldsymbol{y}) = \prod_{i=1}^{\dim} \exp\left[-\left(\frac{\|x_i,y_i\|}{d_i}\right)^{pow_i}\right] \tag{2}$$

in which $d_i (i = 1, \ldots, \dim)$ are positive parameters called *correlation lengths*. With increasing d a stronger statistical correlation of a parameter in space is imposed, and vice versa. The power $pow = 2$ leads to the well-known *bell-shaped* (or Gaussian) autocorrelation function. The combination of identical correlation lengths and *pow*s and use of Euclidean norm imply an isotropic autocorrelation structure. The product structure of autocorrelation (Eq. (2)) is called "fully separable correlation" (a product of correlations in separated dimensions). Typical examples of usage can be found in geology or in space–time contexts, where time is separated.

When the ratio of the measure (size) of Ω to the correlation length d of random field, $r = |\Omega|/d$, is very small (asymptotically approaching zero), the realizations of random fields are constant functions over Ω (correlation coefficients among random variables used for series expansion approach unity). The stochastic field is equivalent to one random variable and can be represented by it. In the opposite direction, when the ratio r is very large (approaching infinity) then the random field fluctuates almost without spatial correlation. This case is then equivalent to the case of large number of statistically independent random variables. In between the ratio r is "reasonable" ($0 \ll r \ll \infty$) and the series expansion methods are random functions which can be suitably simulated by series expansion methods (expansion using a finite number of random variables – coefficients).

Suppose that the spatial variability of a random parameter is described by the Gaussian random field $H(\boldsymbol{x})$, where $\boldsymbol{x}$ is the vector coordinate which determines the position in Ω. Bellow, for the sake of simplicity, we will focus on Karhunen–Loève expansion and an equivalent (approximate) series expansion method (EOLE). These methods are well known in the simulation of univariate random fields and will provide a good basis for illustration of the proposed methodology for the simulation of multivariate random fields.

2.1. Karhunen–Loève expansion (KLE)

The Karhunen–Loève expansion of a stochastic field $H(\boldsymbol{x}, \theta)$ is based on the *spectral expansion* of its autocovariance function $C_{HH}(\boldsymbol{x}, \boldsymbol{y})$. This method is also known as the proper orthogonal decomposition. The autocovariance function being symmetric, bounded and positive definite, by definition, has all its eigenfunctions $\psi_j(\boldsymbol{x})$ mutually orthogonal, and they form a complete set spanning the space to which $H(\boldsymbol{x}, \theta)$ belongs ($\mathscr{L}^2(\Omega)$). The expansion of $H(\boldsymbol{x}, \theta)$ with this deterministic set utilizes orthogonal random coefficients (uncorrelated random variables with zero mean and unit variance) $\xi_j(\theta)$ in the following form:

$$H(\boldsymbol{x}, \theta) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \xi_j(\theta) \psi_j(\boldsymbol{x}) \tag{3}$$

where $\{\lambda_j, \psi_j(\boldsymbol{x})\}$ are the eigenvalues with eigenfunctions of the covariance kernel, a solution of the eigenvalue problem (Fredholm integral equation of the second kind, homogeneous):

$$\int_{\Omega} C_{HH}(\boldsymbol{x}, \boldsymbol{y}) \psi_j(\boldsymbol{y}) \, \mathrm{d}\Omega_y - \lambda_j \psi_j(\boldsymbol{x}) = 0 \tag{4}$$

The analytical solution of the real, positive and numerable spectrum of eigenvalues and associated eigenfunctions is known only for several covariance kernels; however, a numerical solution is available, see e.g. the book by Ghanem and Spanos [16]. We remark that an efficient approximate solution of the eigenproblem can be obtained using the Fast Multipole Method (FMM) first introduced by Rokhlin [43]. FMM is a fast solver of integral equations (see [34] for a review) Application of generalized FMM to KLE approximation has recently been published by Schwab and Todor [45]. Another efficient way of solving the related Fredholm integral equation is the wavelet-Galerkin approach proposed by Phoon et al. [41].

The domain of an eigenvalue problem may not be the domain Ω at which the random field is targeted. To limit variance error at the boundaries, a larger domain may be defined [49].

If the expanded random field $H(\cdot)$ is Gaussian, then the random variables ξ_j form an orthonormal Gaussian vector. Karhunen–Loève expansion [30] is mean-square convergent irrespective of the probabilistic structure of the field being expanded (provided it has a finite variance).

The expansion of any realization of the original random field is done over a complete set of deterministic functions. Practically, the approximation of a field is performed by truncating the series after a finite number of terms (Eq. (5)). It can be shown that the eigenvalues λ_j of the covariance kernel decay monotonically with

the increasing value of its index with the rate related to the rate of decay of the Fourier transform of the autocorrelation function of the expanded field/process. Due to the fact that eigenvalues do not accumulate around a nonzero value, it is possible to order them in a descending series converging to zero. Truncating the ordered series (Eq. (3)) after the N_{var}-th term gives the approximation of the expanded field:

$$\hat{H}(\boldsymbol{x},\theta) = \hat{H}_{N_{\text{var}}} = \sum_{j=1}^{N_{\text{var}}} \sqrt{\lambda_j}\xi_j(\theta)\psi_j(\boldsymbol{x}) \tag{5}$$

The series of approximations $\hat{H}_{N_{\text{var}}}$ converge in the mean square to the original process H. This implies that, for sufficiently large N_{var}, the second-moment properties of $H(\boldsymbol{x})$ can be approximated by the second-moment properties of its parametric representation $\hat{H}_{N_{\text{var}}}$. The number N_{var} to be chosen strongly depends on the desired accuracy and on the autocovariance function of the stochastic field. The mean square error integrated over Ω resulting from a truncation is minimized, e.g. [63]. It is worth pointing that KLE is the most efficient expansion of random processes in terms of the truncated expansion series (i.e. number of random variables N_{var}). The closer a process is to white noise, the more terms are required with a given level of underestimation of variability (which always occurs). An appropriate measure of captured variability is the ratio $c_{N_{\text{var}}} = \sum_{j=1}^{N_{\text{var}}}\lambda_j / \sum_{j=1}^{\infty}\lambda_j$, which can be used for as a correction of each eigenvalue ($c_{N_{\text{var}}}\lambda_j, j = 1 \ldots N_{\text{var}}$) in order to reduce (shift) the variance error along the expanded field over Ω (standardized Karhunen–Loève expansion). Both the variance over Ω and the covariance profile (distance vs. covariance) estimated for a random field simulated by KLE with adjusted eigenvalues (standardized) are closer to the target values than without the standardization. The partial sum of the j largest eigenvalues divided by the total sum (portion of normalized variability) can be continuously plotted vs. k (number of random variables planned for representation, see Fig. 2). The reduction of a number of random variables resulting from such a truncation has a significant impact on the computational cost of the simulation of a random field. It will be shown later that this reduction will be amplified in the proposed simulation of cross correlated stochastic fields.

KLE is one of the basic methods representing series expansion methods with orthogonal functions; several other methods are in fact approximations of KLE. For example, in the orthogonal series expansion method (OSE) proposed by Zhang and Ellingwood [63], the eigenvalue problem (Eq. (4)) is solved numerically by selecting ab initio a complete set of orthogonal functions $h_j(\boldsymbol{x})$. If correlated Gaussian random variables are transformed into uncorrelated variables (random coefficients of functions) using spectral decomposition of the correlation matrix, it can be shown that OSE is an approximation of KLE. The OSE method using a complete set of orthogonal functions $\{h_j(\boldsymbol{x})\}_{j=1}^{\infty}$ is strictly equivalent to KLE in the case that the eigenfunctions $\psi_j(\boldsymbol{x})$ of C_{HH} are approximated using the same set $\{h_j(\boldsymbol{x})\}_{j=1}^{\infty}$.

2.2. Expansion optimal linear estimation (EOLE)

Another representation with a slightly different flavor aims at finding the optimal representation of a stochastic field as a linear combination of a subset of its values that has been sampled over a countable domain from its index space (optimal grid of nodes). This representation was proposed by Li and Der Kiureghian [26]. It is an extension of the OLE method proposed by the same authors (it uses a spectral representation of the nodal values), and which is sometimes referred to as the *Kriging method*, see e.g. Vanmarcke [50]. In fact, the difference is that the Kriging method is usually based on observed nodal values of the field. In the OLE method the field is represented by a linear function of discrete (nodal) field values $\mathbf{u} = \{H(\boldsymbol{x}_1), \ldots, H(\boldsymbol{x}_N)\}$ – a special case of regression on linear functionals [14]. Minimization of the variance error between the target random field and an approximated field $\mathrm{Var}[H(\boldsymbol{x}) - \tilde{H}(\boldsymbol{x})]$ under the condition of equal mean values of both ($E[H(\boldsymbol{x}) - \tilde{H}(\boldsymbol{x})] = 0$) results in [26]:

$$\hat{H}(\boldsymbol{x},\theta) = \Sigma_{H(\boldsymbol{x})\mathbf{u}}^{\mathrm{T}}\Sigma_{\mathbf{uu}}^{-1}(\mathbf{u} - \boldsymbol{\mu}_{\boldsymbol{u}}), \quad \boldsymbol{x} \in \Omega \tag{6}$$

where $\Sigma_{H(\boldsymbol{x})\mathbf{u}}$ is a covariance matrix between the field values $\mathbf{u}$ and an arbitrary point $\boldsymbol{x} \in \Omega$, $\Sigma_{\mathbf{uu}}$ is the covariance matrix of the field values $\mathbf{u}$ and $\boldsymbol{\mu}_{\boldsymbol{u}}$ denotes the mean value of the field over its nodal values $\mathbf{u}$. Among all linear representations of $H(\boldsymbol{x})$ in terms of discrete random variables $\mathbf{u}$ that match the mean function, OLE representation is optimal in the sense that it minimizes the variance error at any given point. This is particularly

desirable for the Gaussian distribution, which is completely defined through the mean value and the variance. Using a spectral representation of the nodal variables (vector $\mathbf{u}$) the OLE method results in the EOLE method [26]:

$$\hat{H}(\boldsymbol{x},\theta) = \sum_{j=1}^{N_{\text{var}}} \frac{\xi_j(\theta)}{\sqrt{\lambda_j^{\mathbf{u}}}} [\boldsymbol{\Phi}_j^{\mathbf{u}}]^{\mathrm{T}} \Sigma_{H(\boldsymbol{x})\mathbf{u}} \tag{7}$$

where $N_{\text{var}} \leqslant N$ represents the truncation in the discrete spectral representation of the field (random vector):

$$\mathbf{H}_{\mathbf{u}}(\theta) = \sum_{j=1}^{N} \xi_j(\theta) \sqrt{\lambda_j^{\mathbf{u}}} [\boldsymbol{\Phi}_j^{\mathbf{u}}]^{\mathrm{T}} \tag{8}$$

where $\lambda_j^{\mathbf{u}}$ and $\boldsymbol{\Phi}^{\mathbf{u}}{}_j$ are the solutions of the eigenvalue problem: $\Sigma_{\mathbf{uu}} \boldsymbol{\Phi}_j^{\mathbf{u}} = \lambda_j^{\mathbf{u}} \boldsymbol{\Phi}_j^{\mathbf{u}}$.

In the selection of an optimal grid a factor is the numerical stability of the probabilistic transformation. If the grid is excessively fine, the grid points are highly correlated and their correlation matrix is nearly singular. Therefore, the transformation may become numerically unstable. From the standpoint of accuracy, one should use a fine grid so the field can then be well represented. The selection of grid density depends on the type of autocorrelation function and length. There are several papers suggesting the number of grid points per autocorrelation length, an extensive source of information with references is the work by Sudret and Der Kiureghian [49].

In the context of the EOLE method we would like to emphasize that an optimal grid of nodal points may be in the one-dimensional case a grid of equidistant points in Ω. The advantage is that the correlation matrix $\Sigma_{\mathbf{uu}}$ becomes a symmetric *Toeplitz matrix* – $N \cdot N$ symmetric matrix whose (i,j)th element is $t_{ij} = t(|i-j|)$, i.e. each element depends only on the difference $|i-j|$. In more dimensions, optimal node numbering can make the $\Sigma_{\mathbf{uu}}$ matrix close to the Toeplitz matrix. Numerical problems involving Toeplitz matrices typically have a fast solution. Having the matrix $\Sigma_{\mathbf{uu}}$ assembled as the symmetric and positive Toeplitz matrix, one can use efficient (specialized) algorithms for spectral decomposition, see e.g. [1,22,7]. In addition to this, the maximum and minimum eigenvalues can also be accurately estimated at a very low cost. This can significantly reduce the computational effort for field expansion.

3. Cross correlated Gaussian random fields

It is usual that more than one random property governs the evolution of a system. Consider for instance Young's modulus, Poisson's ratio or strength in mechanical problems, etc. In a probabilistic concept, all these quantities can be modeled by random fields. There are several papers dealing with the simulation of multivariate random fields using the cross spectral density matrix e.g. [46]. A more recent paper based on spectral representation methodology confirms such stochastic processes as being ergodic [10]. A nice review of the available methods together with a novel hybrid spectral representation and the proper orthogonal decomposition approach for simulation of multivariate Gaussian processes has recently been published by Chen and Letchford [5].

The present paper aims at the utilization of methods based on modal decomposition of correlation matrices. For this case, the simulation of multivariate processes was described by Yamazaki and Shinozuka [62] who used the covariance decomposition method with statistical preconditioning. Their method is a simple extension of the modal decomposition method of the autocovariance matrix that is used in simulation of a univariate random field. The present paper deals with cases when all fields simulated over a region Ω share an identical autocorrelation function over Ω, and the cross correlation structure between each pair of simulated fields is simply defined by a cross correlation coefficient. Such an assumption enables one to perform the modal transformation in two "small" steps, not in one "big" step, as proposed by Yamazaki and Shinozuka [62]. The advantage is a significant reduction in the dimension of the eigenvalue problem considering the fact that the modal decomposition of the given autocorrelation function (KLE) or matrix (EOLE) is done only once. An illustration of the algorithms of both methods and their comparison with a detailed description follow.

The key idea of the proposed method is that all cross correlated fields are expanded using the same spectrum of eigenfunctions/vectors (described in the preceding section for KLE and EOLE methods), but the sets of random variables used for the expansion of each field are cross correlated. In other words, each field is expanded using a set of independent random variables, but these sets must be correlated with respect to the cross correlation matrix among all expanded random fields.

Note that sometimes authors of technical papers utilize cross correlated fields which are independent. However, this independence is only assumed, and the fields are simply simulated one-by-one without any regard given to what the relationship between the sets of representing random variables is. Therefore, some undesired spurious cross correlation can arise among random fields.

3.1. Method for simulation of cross correlated random fields

In this section, we present some definitions needed for the problem formulation, notations and basic facts used throughout the paper. The most important properties of defined items are stated.

D 1 (*Cross correlation matrix of random fields*). Let $\mathbf{C}$ be a square symmetric positive definite matrix of order N_F with elements $C^{i,j} \in (-1;1)$ for $i \neq j$ and $C^{i,j} = 1$ for $i = j$. Matrix $\mathbf{C}$ is a cross correlation matrix and defines the correlation structure among N_F random fields.

The cross correlation matrix $\mathbf{C}$ has N_F real, positive eigenvalues $\lambda_j^{\mathbf{C}},\ j =, 1 \ldots, N_F$ associated with N_F orthonormal eigenvectors $\mathbf{\Phi}^{\mathbf{C}}{}_j, j = 1, \ldots, N_F$. After ordering them so that $\lambda_1^{\mathbf{C}} \geqslant \lambda_2^{\mathbf{C}} \geqslant \ldots \geqslant \lambda_{N_F}^{\mathbf{C}}$ the eigenvector matrix reads:

$$\mathbf{\Phi}^{\mathbf{C}} = \begin{array}{c} \begin{array}{ccccccc} \mathbf{\Phi}_1^{\mathbf{C}} & \mathbf{\Phi}_2^{\mathbf{C}} & \ldots & \mathbf{\Phi}_{N_{F,r}}^{\mathbf{C}} & \ldots & \mathbf{\Phi}_{N_F}^{\mathbf{C}} \end{array} \\ \begin{pmatrix} \phi_{1,1}^{\mathbf{C}} & \phi_{1,2}^{\mathbf{C}} & \ldots & \phi_{1,N_{F,r}}^{\mathbf{C}} & \ldots & \phi_{1,N_F}^{\mathbf{C}} \\ \phi_{2,1}^{\mathbf{C}} & \phi_{2,2}^{\mathbf{C}} & \ldots & \phi_{2,N_{F,r}}^{\mathbf{C}} & \ldots & \phi_{2,N_F}^{\mathbf{C}} \\ \vdots & \vdots & \ldots & \ldots & \ldots & \vdots \\ \phi_{N_F,1}^{\mathbf{C}} & \phi_{N_F,2}^{\mathbf{C}} & \ldots & \phi_{N_F,N_{F,r}}^{\mathbf{C}} & \ldots & \phi_{N_F,N_F}^{\mathbf{C}} \end{pmatrix} \end{array} \tag{9}$$

and

$$\mathbf{\Lambda}^{\mathbf{C}} = diag \begin{array}{c} \begin{array}{ccccccc} \mathbf{\Phi}_1^{\mathbf{C}} & \mathbf{\Phi}_2^{\mathbf{C}} & \ldots & \mathbf{\Phi}_{N_{F,r}}^{\mathbf{C}} & \ldots & \mathbf{\Phi}_{N_F}^{\mathbf{C}} \end{array} \\ \begin{pmatrix} \lambda_1^{\mathbf{C}} & \lambda_2^{\mathbf{C}} & \ldots & \lambda_{N_{F,r}}^{\mathbf{C}} & \ldots & \lambda_{N_F}^{\mathbf{C}} \end{pmatrix} \end{array} \tag{10}$$

Each jth eigenvector $\mathbf{\Phi}_j^{\mathbf{C}}$ is normalized to have an Euclidean length of 1, therefore $[\mathbf{\Phi}^{\mathbf{C}}]^{\mathrm{T}}\mathbf{\Phi}^{\mathbf{C}} = \mathbf{I}$, in which $\mathbf{I}$ is an identity matrix. The spectral decomposition of correlation matrix $\mathbf{C}$ reads: $\mathbf{C}\mathbf{\Phi}^{\mathbf{C}} = \mathbf{\Phi}^{\mathbf{C}}\mathbf{\Lambda}^{\mathbf{C}}$. Let us denote $\mathbf{\Phi}^{\mathbf{C}} = (\mathbf{\Phi}_{\mathrm{I}}^{\mathbf{C}}\ \mathbf{\Phi}_{\mathrm{II}}^{\mathbf{C}})$ and $\mathbf{\Lambda}^{\mathbf{C}} = (\mathbf{\Lambda}_{\mathrm{I}}^{\mathbf{C}}\ \mathbf{\Lambda}_{\mathrm{II}}^{\mathbf{C}})$, where $\mathbf{\Phi}_{\mathrm{I}}^{\mathbf{C}} = (\mathbf{\Phi}_1^{\mathbf{C}}\mathbf{\Phi}_2^{\mathbf{C}} \cdots \mathbf{\Phi}_{N_{F,r}}^{\mathbf{C}})$ is the $(N_F \times N_{F,r})$ matrix and $\mathbf{\Lambda}_{\mathrm{I}}^{\mathbf{C}} = \mathrm{diag}(\lambda_1^{\mathbf{C}}, \ldots, \lambda_{N_{F,r}}^{\mathbf{C}})$ is the $(N_{F,r} \times N_{F,r})$ diagonal matrix. Then, by partitioning the matrices $\mathbf{\Phi}^{\mathbf{C}}$ and $\mathbf{\Lambda}^{\mathbf{C}}$ the spectral decomposition can be written as:

$$\mathbf{C}\begin{pmatrix} \mathbf{\Phi}_{\mathrm{I}}^{\mathbf{C}} & \mathbf{\Phi}_{\mathrm{II}}^{\mathbf{C}} \end{pmatrix} = \begin{pmatrix} \mathbf{\Phi}_{\mathrm{I}}^{\mathbf{C}} & \mathbf{\Phi}_{\mathrm{II}}^{\mathbf{C}} \end{pmatrix} \begin{pmatrix} \mathbf{\Lambda}_{\mathrm{I}}^{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_{\mathrm{II}}^{\mathbf{C}} \end{pmatrix} \tag{11}$$

Partitioning of the matrices will be used later in the reduction of computational effort for the simulation of random fields. It can be shown that a large amount of computer memory can be saved at a given level of accuracy if one uses $\mathbf{\Phi}_{\mathrm{I}}^{\mathbf{C}}$ instead of full $\mathbf{\Phi}^{\mathbf{C}}$ (with associated $\mathbf{\Lambda}'$s). The idea is that the largest eigenvalues and their corresponding eigenvectors dominate the foregoing transformation, so the second part of the eigenvalues/vectors can be neglected and the *approximate spectral representation of matrix* $\widehat{\mathbf{C}}$ can be obtained:

$$\widehat{\mathbf{C}} = \mathbf{\Phi}_{\mathrm{I}}^{\mathbf{C}}\mathbf{\Lambda}_{\mathrm{I}}^{\mathbf{C}}[\mathbf{\Phi}_{\mathrm{I}}^{\mathbf{C}}]^{\mathrm{T}} \tag{12}$$

As a measure of an error introduced by the reduction $N_F \to N_{F,r}$, the following can serve a ratio based on the portion of the largest eigenvalues considered:

$$\varepsilon_C = \frac{\sum_{j=1}^{N_{F,r}} \lambda_j^{\mathbf{C}}}{\sum_{j=1}^{N_F} \lambda_j^{\mathbf{C}}} \tag{13}$$

It can be shown that for the simulation of cross correlated stochastic fields by the methods described above one needs to simulate a vector of cross correlated random variables for the expansion. These random variables have a correlation matrix defined as follows.

D 2 (*Block cross correlation matrix* ***D*** *of random variables*). Let **D** be a squared symmetric matrix of order $(N_F N_{\text{var}})$ assembled in this way: matrix **D** consists of $(N_F \times N_F)$ blocks (squared matrices) $C^{i,j}\mathbf{I}$, where **I** is the unit matrix of order N_{var}, and $C^{i,j}$ are elements of the cross correlation matrix **C** defined previously

$$\mathbf{D} = \begin{array}{c} \\ H_1 \\ H_2 \\ H_3 \\ \vdots \\ H_{N_F} \end{array} \begin{array}{c} \begin{array}{ccccc} H_1 & H_2 & H_3 & \dots & H_{N_F} \end{array} \\ \begin{pmatrix} \mathbf{I} & C^{1,2}\mathbf{I} & C^{1,3}\mathbf{I} & \dots & C^{1,N_F}\mathbf{I} \\ \vdots & \mathbf{I} & C^{2,3}\mathbf{I} & \dots & C^{2,N_F}\mathbf{I} \\ \vdots & \vdots & \mathbf{I} & \dots & C^{3,N_F}\mathbf{I} \\ \vdots & sym. & \vdots & \ddots & \vdots \\ \dots & \dots & \dots & \dots & \mathbf{I} \end{pmatrix} \end{array}$$

D is a correlation matrix having nonzero elements on sub-diagonals of partial square blocks. The fact that each square block matrix on the diagonal of **D** is the $(N_{\text{var}} \times N_{\text{var}})$ unit matrix can be simply interpreted: random variables needed for the expansion of one random field H_i, $i = 1, \dots, N_F$ are non-correlated (and also independent since we will work with Gaussian random variables). The off-diagonal square blocks (diagonal matrices) represent cross correlation between each two sets of random variables used for expansion of the fields H_i and H_j, $i \neq j$; $i, j = 1, \dots, N_F$. The key property for the proposed method is the spectral property of the correlation matrix **D**. Cross correlation matrix **D** has $N_F N_{\text{var}}$ real, positive eigenvalues $\lambda_j^{\mathbf{D}}$, $j = 1, \dots, (N_F N_{\text{var}})$ associated with orthogonal eigenvectors. Obviously matrix **D** has the same eigenvalues as matrix **C**, but these are N_{var}-multiple. Similarly the eigenvectors of **D** are associated with the eigenvectors of **C**. The space described by $\mathbf{\Phi}^{\mathbf{C}}$ is enriched so that the dimension is N_{var}-times higher, but the components of the orthogonal eigenvectors $\mathbf{\Phi}^{\mathbf{C}}$ remain.

After ordering the eigenvalues so that $\lambda_1^{\mathbf{D}} \geqslant \lambda_2^{\mathbf{D}} \geqslant \cdots \geqslant \lambda_{N_F \cdot N_{\text{var}}}^{\mathbf{D}}$, one can assemble the eigenvectors/eigenvalue matrices using a block-matrix with squared block submatrices:

$$\mathbf{\Phi}^{\mathbf{D}} = \begin{pmatrix} \mathbf{\Phi}_1^{\mathbf{D}} & \mathbf{\Phi}_2^{\mathbf{D}} & \dots & \mathbf{\Phi}_{N_{F,r}}^{\mathbf{D}} & \dots & \mathbf{\Phi}_{N_F}^{\mathbf{D}} \\ \phi_{1,1}^{\mathbf{C}}\mathbf{I} & \phi_{1,2}^{\mathbf{C}}\mathbf{I} & \dots & \phi_{1,N_{F,r}}^{\mathbf{C}}\mathbf{I} & \dots & \phi_{1,N_F}^{\mathbf{C}}\mathbf{I} \\ \phi_{2,1}^{\mathbf{C}}\mathbf{I} & \phi_{2,2}^{\mathbf{C}}\mathbf{I} & \dots & \phi_{2,N_{F,r}}^{\mathbf{C}}\mathbf{I} & \dots & \phi_{2,N_F}^{\mathbf{C}}\mathbf{I} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ \phi_{N_F,1}^{\mathbf{C}}\mathbf{I} & \phi_{N_F,2}^{\mathbf{C}}\mathbf{I} & \dots & \phi_{N_F,NF,r}^{\mathbf{C}}\mathbf{I} & \dots & \phi_{N_F,N_F}^{\mathbf{C}}\mathbf{I} \end{pmatrix} \tag{14}$$

and the eigenvalue matrices corresponding to vector blocks $(\mathbf{\Phi}_1^{\mathbf{D}}, \dots, \mathbf{\Phi}_{N_F}^{\mathbf{D}})$:

$$\mathbf{\Lambda}^{\mathbf{D}} = diag \begin{array}{c} \begin{array}{cccccc} \mathbf{\Phi}_1^{\mathbf{D}} & \mathbf{\Phi}_2^{\mathbf{D}} & \dots & \mathbf{\Phi}_{N_{F,r}}^{\mathbf{D}} & \dots & \mathbf{\Phi}_{N_F}^{\mathbf{D}} \end{array} \\ \begin{pmatrix} \lambda_1^{\mathbf{C}}\mathbf{I} & \lambda_2^{\mathbf{C}}\mathbf{I} & \dots & \lambda_{N_{F,r}}^{\mathbf{C}}\mathbf{I} & \dots & \lambda_{N_F}^{\mathbf{C}}\mathbf{I} \end{pmatrix} \end{array} \tag{15}$$

where **I** is the unit matrix of order N_{var}. The spectral decomposition of correlation matrix **D** reads: $\mathbf{D}\mathbf{\Phi}^{\mathbf{D}} = \mathbf{\Phi}^{\mathbf{D}}\mathbf{\Lambda}^{\mathbf{D}}$. Let us denote $\mathbf{\Phi}^{\mathbf{D}} = (\mathbf{\Phi}_{\text{I}}^{\mathbf{D}} \ \mathbf{\Phi}_{\text{II}}^{\mathbf{D}})$ and $\mathbf{\Lambda}^{\mathbf{D}} = (\mathbf{\Lambda}_{\text{I}}^{\mathbf{D}} \ \mathbf{\Lambda}_{\text{II}}^{\mathbf{D}})$; $\mathbf{\Phi}_{\text{I}}^{\mathbf{D}} = (\mathbf{\Phi}_1^{\mathbf{D}} \dots \mathbf{\Phi}_{N_{F,r}}^{\mathbf{D}})$ and $\mathbf{\Lambda}_{\text{I}}^{\mathbf{D}} = \text{diag}(\lambda_1^{\mathbf{C}}\mathbf{I}, \dots, \lambda_{N_{F,r}}^{\mathbf{C}}\mathbf{I})$. Then by partitioning the matrices $\mathbf{\Phi}^{\mathbf{D}}$ and $\mathbf{\Lambda}^{\mathbf{D}}$ the spectral decomposition can be written as:

$$\mathbf{D}\begin{pmatrix} \mathbf{\Phi}_{\text{I}}^{\mathbf{D}} & \mathbf{\Phi}_{\text{II}}^{\mathbf{D}} \end{pmatrix} = \begin{pmatrix} \mathbf{\Phi}_{\text{I}}^{\mathbf{D}} & \mathbf{\Phi}_{\text{II}}^{\mathbf{D}} \end{pmatrix} \begin{pmatrix} \mathbf{\Lambda}_{\text{I}}^{\mathbf{D}} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_{\text{II}}^{\mathbf{D}} \end{pmatrix} \tag{16}$$

Matrices **C** and **D** are positive definite. Similarly to Eq. (12) the second part of the eigenvalues/vectors can be neglected and the *approximate spectral representation of (cross) correlation matrix* $\widehat{\mathbf{D}}$ can be obtained as:

$$\widehat{\mathbf{D}} = \mathbf{\Phi}_{\mathrm{I}}^{\mathbf{D}} \mathbf{\Lambda}_{\mathrm{I}}^{\mathbf{D}} [\mathbf{\Phi}_{\mathrm{I}}^{\mathbf{D}}]^{\mathrm{T}} \tag{17}$$

where the matrix $\mathbf{\Phi}_{\mathrm{I}}^{\mathbf{D}}$ contains only the respective eigenvectors to the $N_{F,r}$ eigenvalues.

The reduction of the eigenvectors/eigenvalues used in Eq. (17) is "quantized", because each eigenvector submatrix $\mathbf{\Phi}_j^{\mathbf{D}}$, $j = 1, \ldots, N_F$ consists of N_{var} orthogonal eigenvectors. The same applies to eigenvalue matrix $\mathbf{\Lambda}^{\mathbf{D}}$. The utilization of each next eigenvector of the "source" correlation matrix $\mathbf{C}$ results in the increase of N_{var} eigenvectors of matrix $\mathbf{D}$ in Eq. (17). An error introduced by reduction of the number of eigenmodes considered is analogous to that in Eq. (13):

$$\varepsilon_D = \frac{\sum_{j=1}^{N_r} \lambda_j^{\mathbf{D}}}{\sum_{j=1}^{N_{\mathrm{var}} \cdot N_F} \lambda_j^{\mathbf{D}}} = \frac{\sum_{j=1}^{N_{F,r}} (N_{\mathrm{var}} \lambda_j^{\mathbf{C}})}{\sum_{j=1}^{N_F} (N_{\mathrm{var}} \lambda_j^{\mathbf{C}})} = \frac{\sum_{j=1}^{N_{F,r}} \lambda_j^{\mathbf{C}}}{\sum_{j=1}^{N_F} \lambda_j^{\mathbf{C}}} = \varepsilon_C \tag{18}$$

in which $N_r = N_{\mathrm{var}} N_{F,r}$ is the reduced number of eigenmodes considered; it will be shown later that N_r is also the number of random variables that must be simulated to represent a vector random field. It might be important to know how the correlation matrix of all N_F fields, each discretized into the same set of N points $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$, looks:

D 3 (*Full-block correlation matrix* $\boldsymbol{F}$). Let $\mathbf{F}$ be a squared symmetric matrix of order $N_F N$ assembled as follows. Matrix $\mathbf{F}$ consists of $N_F \times N_F$ blocks (squared matrices) $\mathbf{F}_{\mathbf{uu}}^{i,j}$ which are correlation matrices of order N. Each submatrix $\mathbf{F}_{\mathbf{uu}}^{i,j}$ is symmetric:

$$\mathbf{F}_{\mathbf{uu}}^{i,j} = \mathbf{F}_{\mathbf{uu}}^{j,i} = \begin{array}{c} \\ \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_k \\ \vdots \\ \boldsymbol{x}_N \end{array} \begin{array}{c} \begin{array}{cccccc} \boldsymbol{x}_1 & \ldots & \boldsymbol{x}_l & \ldots & \boldsymbol{x}_N \end{array} \\ \begin{pmatrix} F_{1,1}^{i,j} & \ldots & F_{1,l}^{i,j} & \ldots & F_{1,N}^{i,j} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ F_{k,1}^{i,j} & \ldots & F_{k,l}^{i,j} & \ldots & F_{k,N}^{i,j} \\ \vdots & \ldots & \vdots & \ddots & \vdots \\ F_{N,1}^{i,j} & \ldots & F_{N,l}^{i,j} & \ldots & F_{N,N}^{i,j} \end{pmatrix} \end{array} \tag{19}$$

and the general entry $F_{k,l}^{i,j} = F_{l,k}^{i,j} = \mathrm{Corr}[H_i(x_k), H_j(x_l)]$ has the meaning of correlation between two field's (i,j) nodal values at points x_k, x_l $(k, l = 1, \ldots, N)$. Matrix $\mathbf{F}$ can be obtained using the autocorrelation matrix $\mathbf{F}_{\mathbf{uu}} = \mathbf{F}_{\mathbf{uu}}^{i,i}$ and using the cross correlation matrix $\mathbf{C}$ among random fields (vectors) $\mathbf{H}_1, \ldots, \mathbf{H}_{N_F}$ simply by multiplying the autocorrelation by the cross correlation: $F_{k,l}^{i,j} = C^{i,j} F_{k,l}$. Matrix $\mathbf{F}$ can be written using the squared $(N \times N)$ blocks $\mathbf{F}_{\mathbf{uu}}^{i,j} = C^{i,j} \mathbf{F}_{\mathbf{uu}}$ as:

$$\mathbf{F} = \begin{array}{c} \\ \mathbf{H}_1 \\ \mathbf{H}_2 \\ \vdots \\ \mathbf{H}_{N_F} \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{H}_1 & \mathbf{H}_2 & \ldots & \mathbf{H}_{N_F} \end{array} \\ \begin{pmatrix} \mathbf{F}_{\mathbf{uu}} & C^{1,2}\,\mathbf{F}_{\mathbf{uu}} & \ldots & C^{1,N_F}\,\mathbf{F}_{\mathbf{uu}} \\ \vdots & \mathbf{F}_{\mathbf{uu}} & \ldots & C^{2,N_F}\,\mathbf{F}_{\mathbf{uu}} \\ \vdots & sym. & \ddots & \vdots \\ \ldots & \ldots & \ldots & \mathbf{F}_{\mathbf{uu}} \end{pmatrix} \end{array} \tag{20}$$

This illustrates the simple cross correlation relationships between the vector fields $\mathbf{H}_i, \mathbf{H}_j$ (single correlation coefficients $C^{i,j}$).

Matrix $\mathbf{F}$ is the target cross correlation matrix of discretized random fields (random vectors) $\mathbf{H}_1, \ldots, \mathbf{H}_{N_F}$, each discretized into the same set of points $\boldsymbol{x}_i$ $(i = 1, \ldots, N)$.

It is not difficult to show that if the correlation matrix $\mathbf{F}$ consists of blocks (autocorrelation matrices $\mathbf{F}_{\mathbf{uu}}$, each multiplied by a cross correlation coefficient $C^{i,j}$), the eigenvector matrix denoted $\mathbf{\Phi}^{\mathbf{u}}$ can be assembled as a block-matrix with block submatrices $\mathbf{\Phi}_1^{\mathbf{F}}, \ldots, \mathbf{\Phi}_{N_F}^{\mathbf{F}}$:

$$\boldsymbol{\Phi}^{\mathbf{F}} = \begin{array}{c} \begin{array}{ccccccc} \boldsymbol{\Phi}^{\mathbf{F}}_1 & \boldsymbol{\Phi}^{\mathbf{F}}_2 & \dots & \boldsymbol{\Phi}^{\mathbf{F}}_{N_{F,r}} & \dots & \boldsymbol{\Phi}^{\mathbf{F}}_{N_F} \end{array} \\ \begin{pmatrix} \phi^{\mathbf{C}}_{1,1}\boldsymbol{\Phi}^{\mathbf{u}} & \phi^{\mathbf{C}}_{1,2}\boldsymbol{\Phi}^{\mathbf{u}} & \dots & \phi^{\mathbf{C}}_{1,N_{F,r}}\boldsymbol{\Phi}^{\mathbf{u}} & \dots & \phi^{\mathbf{C}}_{1,N_F}\boldsymbol{\Phi}^{\mathbf{u}} \\ \phi^{\mathbf{C}}_{2,1}\boldsymbol{\Phi}^{\mathbf{u}} & \phi^{\mathbf{C}}_{2,2}\boldsymbol{\Phi}^{\mathbf{u}} & \dots & \phi^{\mathbf{C}}_{2,N_{F,r}}\boldsymbol{\Phi}^{\mathbf{u}} & \dots & \phi^{\mathbf{C}}_{2,N_F}\boldsymbol{\Phi}^{\mathbf{u}} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ \phi^{\mathbf{C}}_{N_F,1}\boldsymbol{\Phi}^{\mathbf{u}} & \phi^{\mathbf{C}}_{N_F,2}\boldsymbol{\Phi}^{\mathbf{u}} & \dots & \phi^{\mathbf{C}}_{N_F,NF,r}\boldsymbol{\Phi}^{\mathbf{u}} & \dots & \phi^{\mathbf{C}}_{N_F,N_F}\boldsymbol{\Phi}^{\mathbf{u}} \end{pmatrix} \end{array} \tag{21}$$

and

$$\boldsymbol{\Lambda}^{\mathbf{F}} = diag \begin{array}{c} \begin{array}{ccccccc} \boldsymbol{\Phi}^{\mathbf{F}}_1 & \boldsymbol{\Phi}^{\mathbf{F}}_2 & \dots & \boldsymbol{\Phi}^{\mathbf{F}}_{N_{F,r}} & \dots & \boldsymbol{\Phi}^{\mathbf{F}}_{N_F} \end{array} \\ \begin{pmatrix} \lambda^{\mathbf{C}}_1\boldsymbol{\Lambda}^{\mathbf{u}} & \lambda^{\mathbf{C}}_2\boldsymbol{\Lambda}^{\mathbf{u}} & \dots & \lambda^{\mathbf{C}}_{N_{F,r}}\boldsymbol{\Lambda}^{\mathbf{u}} & \dots & \lambda^{\mathbf{C}}_{N_F}\boldsymbol{\Lambda}^{\mathbf{u}} \end{pmatrix} \end{array} \tag{22}$$

where $\boldsymbol{\Lambda}^{\mathbf{u}}$ is the (reduced) eigenvalue matrix of $\mathbf{F}$ of order N_{var} and $\lambda^{\mathbf{C}}_i$ $(i = 1,\dots,N_F)$ are the eigenvalues of cross correlation matrix $\mathbf{C}$. Note that the eigenvalues $\lambda^{\mathbf{F}}$ are not sorted automatically even if the eigenvalues of both $\boldsymbol{\Lambda}^{\mathbf{u}}$ and $\boldsymbol{\Lambda}^{\mathbf{C}}$ are sorted. The partitioning of $\boldsymbol{\Phi}^{\mathbf{F}}$ and $\boldsymbol{\Lambda}^{\mathbf{F}}$ in the case that only the reduced number of eigenmodes $N_{F,r}$ of matrix $\mathbf{C}$ are available is obvious.

D 4 (*Block sample matrix* $\chi^{\mathbf{D}}$). Let $\chi^{\mathbf{D}}$ be a $(N_{\text{var}}N_F)$-dimensional jointly normally distributed random vector with correlation matrix $\mathbf{D}$. The vector consists of N_F blocks. Each block (sub-vector) $\boldsymbol{\chi}^{\mathbf{D}}_j$, $j = 1,\dots,N_F$ represents a Gaussian random vector with N_{var} standard Gaussian independent (and therefore also non-correlated) random variables (marginals) while the vectors $\boldsymbol{\chi}^{\mathbf{D}}_i$, $\boldsymbol{\chi}^{\mathbf{D}}_j$ are cross correlated.

For a given number of realizations N_{sim} the vector $\boldsymbol{\chi}^{\mathbf{D}}$ is represented by an $(N_{\text{var}}N_F) \times N_{\text{sim}}$ random matrix. Each of the N_{sim} columns is one realization of a Gaussian random vector. The random vector $\boldsymbol{\chi}^{\mathbf{D}}$ is partitioned into N_F vectors each with the dimension N_{var}:

$$\boldsymbol{\chi}^{\mathbf{D}} = \left[\begin{array}{cccc} [\boldsymbol{\chi}^{\mathbf{D}}_1]^{\mathrm{T}} & [\boldsymbol{\chi}^{\mathbf{D}}_2]^{\mathrm{T}} & [\boldsymbol{\chi}^{\mathbf{D}}_3]^{\mathrm{T}} & \cdots \quad [\boldsymbol{\chi}^{\mathbf{D}}_{N_F}]^{\mathrm{T}} \end{array} \right]^{\mathrm{T}}$$

Simulation of the matrix $\boldsymbol{\chi}^{\mathbf{D}}$ is the most important step in the method. The matrix $\mathbf{D}$ is targeted in simulation of $\boldsymbol{\chi}^{\mathbf{D}}$ as the correlation matrix. The key idea of the method is the utilization of spectral decomposition of correlation matrix $\mathbf{D}$ as this decomposition is very easy to perform (Eq. 14). Therefore, the orthogonal transformation of correlation matrix will be used. The utilization of the equivalence with prescribed correlation matrix $\mathbf{C}$ among fields has a significant computational impact: instead of solving the $N_{\text{var}} \times N_F$ eigenvalue problem of $\mathbf{D}$, one needs to solve the N_F eigenvalue problem of prescribed correlation matrix $\mathbf{C}$. In cases when the number of random variables utilized in the expansion of one random field is large (thousands), the reduction is significant. By partitioning the matrix $\boldsymbol{\chi}^{\mathbf{D}}$ into N_{var}-dimensional blocks, one obtains an independent standard Gaussian random vector for the simulation of each of the N_F random fields.

Having Eq. (14) for the correlation matrix defined in Eq. (2) at hand the simulation of the Block sample matrix $\boldsymbol{\chi}^{\mathbf{D}}$ is straightforward (orthogonal transformation of the correlation matrix):

$$\boldsymbol{\chi}^{\mathbf{D}} = \boldsymbol{\Phi}^{\mathbf{D}}(\boldsymbol{\Lambda}^{\mathbf{D}})^{1/2}\boldsymbol{\xi} \tag{23}$$

where $\boldsymbol{\xi} = \{\xi_i,\ i = 1,\dots,N_F \times N_{\text{var}}\}$ forms a vector of independent standard Gaussian random variables. Employing Eq. (17) for approximate spectral representation, the approximate Block sample matrix $\hat{\boldsymbol{\chi}}^{\mathbf{D}}$ can be simulated similarly:

$$\hat{\boldsymbol{\chi}}^{\mathbf{D}} = \boldsymbol{\Phi}^{\mathbf{D}}_{\mathrm{I}}(\boldsymbol{\Lambda}^{\mathbf{D}}_{\mathrm{I}})^{1/2}\boldsymbol{\xi}_r \tag{24}$$

where $\boldsymbol{\xi}_r = \{\xi_j,\ j = 1,\dots,N_{F,r} \times N_{\text{var}}\}$ again forms a vector of independent standard Gaussian random variables. Matrices $\boldsymbol{\Lambda}^{\mathbf{D}}$ and $\boldsymbol{\Lambda}^{\mathbf{D}}_{\mathrm{I}}$ play the role of (diagonal) covariance matrices.

Because the random variables in vectors $\boldsymbol{\chi}^{\mathbf{D}}$ and $\hat{\boldsymbol{\chi}}^{\mathbf{D}}$ are zero-mean Gaussian random variables, the error between $\boldsymbol{\chi}^{\mathbf{D}}$ and $\hat{\boldsymbol{\chi}}^{\mathbf{D}}$ is defined completely by the difference between the exact correlation matrix $\mathbf{D}$ and the approximate correlation matrix $\widehat{\mathbf{D}}$. The reduction error can be measured by the ratio ε_D (Eq. (18)).

A correlation matrix $\mathbf{C}$ of order N_F has the trace: $\mathrm{tr}(\mathbf{C}) = \sum_{j=1}^{N_F} \lambda_j^{\mathbf{C}} = N_F$. Similarly for a correlation matrix $\mathbf{D}$ of order $N_F N_{\mathrm{var}}$ the trace is $\mathrm{tr}(\mathbf{D}) = \sum_{j=1}^{N_F N_{\mathrm{var}}} \lambda_j^{\mathbf{D}} = N_F N_{\mathrm{var}}$. These properties are important for the error estimation in Eqs. (13), (18) and for the standardization of eigenvalues (see Section 2.1).

Of course, the (sparse) matrices $\mathbf{\Phi}^{\mathbf{D}}$ and $\mathbf{\Lambda}^{\mathbf{D}}$ do not need to be assembled and stored in computer memory. They can be used in the form of an algorithm, and only the eigen-matrices $\mathbf{\Phi}^{\mathbf{C}}$ and $\mathbf{\Lambda}^{\mathbf{C}}$ must be solved (or at least their dominating parts $\mathbf{\Phi}_{\mathrm{I}}^{\mathbf{C}}$ and $\mathbf{\Lambda}_{\mathrm{I}}^{\mathbf{C}}$).

3.2. Simulation of random variables

The most important aspect of the presented linear transformations is that the spatial random fluctuations have been decomposed into a set of deterministic functions in the spatial variables and random coefficients that are independent of these variables. If the random fields $H_i(\boldsymbol{x}, \theta)$ are Gaussian, then Gaussian random variables $\{\xi_j\}$ form an orthonormal Gaussian vector. Once having the deterministic part solved the accuracy of fields depends solely on the quality of random variables. For the purpose of simulation of random vector $\boldsymbol{\xi}$ (ξ_r) plenty of techniques can be used.

In cases when computationally demanding SFEM analysis employs random fields, a suitable type of simulation technique should be applied in order to keep the number of required simulations at an acceptable level (at a given accuracy). One of the alternatives is the Latin Hypercube Sampling (LHS) technique [32,24]. This stratified sampling technique (usually used for simulation of random variables only) has also been recommended for simulation of random variables employed in random fields by many authors [62,36,39]. A more recent comparison of efficiency between the employment of LHS and the Crude Monte Carlo technique for simulation of random fields can be found in [57]. There are two steps in LHS: stratified sampling and handling of the correlation structure of samples. A good improvement to sampling is given in a technical note by Huntington and Lyrintzis [23], who recommend the representative values to be the probabilistic means of each strata:

$$\xi_k = N_{\mathrm{sim}} \int_{b_{k-1}}^{b_k} \xi g(\xi)\,\mathrm{d}\xi, \quad b_k = G^{-1}\left(\frac{k}{N_{\mathrm{sim}}}\right), \tag{25}$$

not the median $\xi_k = G^{-1}((k-0.5)/N_{\mathrm{sim}})$ as is the current practice ($k = 1, \ldots, N_{\mathrm{sim}}$, ξ_k is the kth sample value, $g(\cdot)[G(\cdot)]$ is the probability [cumulative] density function of the sampled random variable, each sample value represents the probability $1/N_{\mathrm{sim}}$ of its strata). There is also an improvement to the correlation structure control published in papers by Vořechovský and Novák [56,58,60]. It will be shown later that such an improvement has a remarkable effect on the accuracy of the autocorrelation properties of simulated random fields.

3.3. Summary of the method

The proposed procedure for the simulation of random fields can be itemized as follows:

(1) The spectral analysis of the autocorrelation structure of expanded field(s). This step is known from the simulation of univariate random fields and is performed once only. At this step the number of random variables needed for expansion of each field N_{var} is determined and the decision about the truncation of eigenvalues is made. For both discretization methods presented in the paper (KLE and EOLE), eigenmodes are recommended to be sought for an enlarged region $\Omega \to \hat{\Omega}$ in order to reduce the variance error at the boundaries. Also, the standardization of N_{var} largest eigenvalues should be performed with regard to the number N_{var} of eigenmodes chosen (see Section 2.1).
   (a) *KLE*: Decomposition of the autocovariance function (Eq. (4)): eigenvalues and eigenfunctions on $\hat{\Omega}$. The number of eigenfunctions is N_{var}.
   (b) *EOLE*: The choice of density of a grid over $\hat{\Omega}$ depending on the autocorrelation function type and correlation lengths must be made. Based on the discretization, the autocorrelation matrix $\mathbf{F_{uu}}$ is assembled and its N_{var} largest eigenvalues and the corresponding eigenvectors are determined.

(2) Eigenvalues must be computed with corresponding orthogonal eigenvectors ($\mathbf{\Phi}_{\mathrm{I}}^{\mathbf{C}} \mathbf{\Lambda}_{\mathrm{I}}^{\mathbf{C}}$) of the target cross correlation matrix $\mathbf{C}$ among random fields. The choice of number of eigenmodes $N_{F,r} \leqslant N_F$ is made.

(3) Simulation of Gaussian random vector $\boldsymbol{\xi}_r$ of $N_r = N_{F,r}N_{\text{var}}$ independent standard Gaussian variables ξ_j. For a given number of simulations N_{sim} a random vector becomes an $N_r \times N_{\text{sim}}$ random matrix, where N_{sim} is a sample size for each random variable. The LHS technique is recommended for the simulation of the random vector (see Section 3.2).

(4) The simulation of cross correlated random vector $\boldsymbol{\chi}^{\mathbf{D}}$ by matrix multiplication (Eq. (23)) or (Eq. (24)). Matrices from Eqs. (14) and (15) of the matrix $\mathbf{D}$ (an enlarged matrix from step 2) and a random matrix from step (3) are utilized.

(5) In this step, the simulation of all (standardized Gaussian) random fields $i = 1,2,\ldots,N_F$ takes place depending on the method from step (1):

(a) *KLE*: Composition of target random functions based on Eq. (5). Each random field $\boldsymbol{i}$ is expanded using the $\boldsymbol{i}$th block $\boldsymbol{\chi}_i^{\mathbf{D}}$ of random vector $\boldsymbol{\chi}^{\mathbf{D}}$ ($\hat{\boldsymbol{\chi}}^{\mathbf{D}}$) and the N_{var} eigenmodes from step (1a):

$$\widehat{H}_i(x) = \sum_{j=1}^{N_{\text{var}}} \sqrt{\lambda_j}\chi_{i,j}^{\mathbf{D}}\psi_j(x) \tag{26}$$

(b) *EOLE*: Similarly, the fields are expanded according to multiplication in Eq. (7):

$$\hat{H}_i(x) = \sum_{j=1}^{N_{\text{var}}} \frac{\chi_{i,j}^{\mathbf{D}}}{\sqrt{\lambda_j^{\mathbf{u}}}}[\boldsymbol{\Phi}_j^{\mathbf{u}}]^{\mathrm{T}}\Sigma_{H(x)\mathbf{u}} \tag{27}$$

(6) The last step is the transformation of standardized Gaussian random field values $i = 1,2,\ldots,N_F$ with respect to target mean values μ_i and standard deviations σ_i via $\hat{H}_i(x)\sigma_i + \mu_i$.

In case of the EOLE method the target correlation matrix of the vector field discretized in the optimal grid is exactly the full-block correlation matrix $\mathbf{F}$.

If statistically independent random fields are required, the cross correlation matrix $\mathbf{C}$ is the identity matrix. Both the eigenvalue and the eigenvector matrix are also the identity matrices. Therefore steps (2) and (4) can be skipped, $\boldsymbol{\xi}_r = \hat{\boldsymbol{\chi}}^{\mathbf{D}}$, $N_{F,r} = N_F$.

In the opposite direction, any covariance matrix $\mathbf{A}$ of order N: $\mathbf{A} = \{A_{i,j} \in \boldsymbol{R} : A_{i,j} = c;\ i,j = 1,\ldots,N\}$ has the first eigenvalue $\lambda_1^{\mathbf{A}} = cN$ and $(\lambda_i^{\mathbf{A}} = 0,\ i = 2,\ldots,N)$. The normalized eigenvector of the first mode is $\boldsymbol{\Phi}_1^{\mathbf{A}} = (N^{-1/2}, N^{-1/2}, \ldots, N^{-1/2})^{\mathrm{T}}$. This means that only the first eigenvalue is meaningful, and hence the problem reduces to: (i) if $\mathbf{C} = \mathbf{A}$, just one random field in the case of cross correlated random fields, (ii) just one random variable for each field if the autocorrelation matrix $\mathbf{F}_{\mathbf{uu}} = \mathbf{A}$ (in the case of an optimal grid in the EOLE method for one random field expansion). The latter case is equivalent to the case of infinite correlation length in the autocovariance kernel in KLE.

Yamazaki and Shinozuka [62] proposed the universal simulation of discretized multivariate stochastic fields by one orthogonal transformation of (block) covariance matrix $\mathbf{F}$. The modal matrix of matrix $\mathbf{F}$ is then used for the transformation of random vector $\boldsymbol{\xi}$ composed of $N \times N_F$ independent Gaussian random variables. The main difference from the method proposed here is that they need to solve an eigenvalue problem of matrix $\mathbf{F}$ that has a large order $(N \times N_F)$ while in this paper the problem is decomposed into two separate modal solutions, namely (i) the autocovariance structure (order N in EOLE; a reduced number of N_{var} eigenmodes must be solved) and (ii) the cross correlation matrix of order N_F ($N_{F,r}$ modes). A simple illustration with a comparison of the approaches is given in Fig. 1. The figure illustrates (a) the expansion of a univariate random field using the random vector $\boldsymbol{\xi}$ and the eigenvalue matrix $\boldsymbol{\Lambda}$ with associated eigenfunctions [eigenvectors] in KLE [EOLE], (b) the simulation procedure employing one "huge" orthogonal transformation of the correlation matrix $\mathbf{F}$ [62]:

$$\mathbf{H} = \boldsymbol{\Phi}^{\mathbf{F}}(\boldsymbol{\Lambda}^{\mathbf{F}})^{1/2}\boldsymbol{\xi} \tag{28}$$

This procedure is general. In our case the correlation matrix $\mathbf{F}$ can be assembled using the products of the cross correlation matrix $\mathbf{C}$ and autocorrelation matrix $\mathbf{F}_{\mathbf{uu}}$. We have shown that the eigenvector and eigenvalue matrices of $\mathbf{C}$ and $\mathbf{F}_{\mathbf{uu}}$ solved separately can be used to compute the required matrices $\boldsymbol{\Phi}^{\mathbf{F}}$ and $\boldsymbol{\Lambda}^{\mathbf{F}}$ (see Eqs. (21), (22)) and therefore computational effort can be saved. It will be shown later that such a technique yields

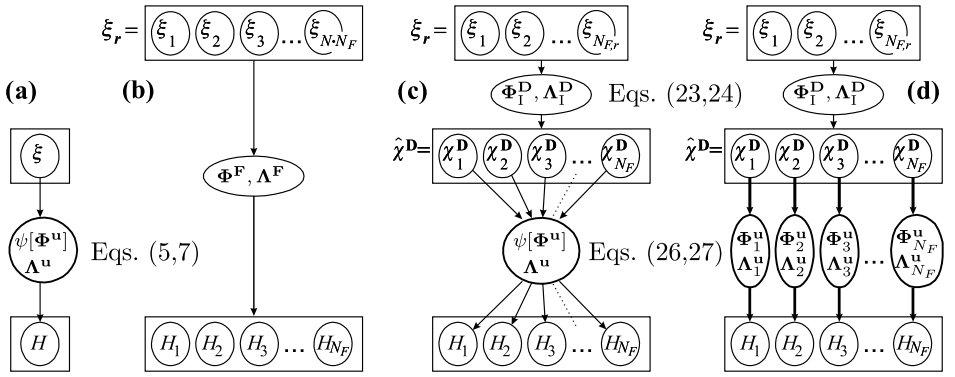

Fig. 1. (a) Simulation of a univariate random field using N_{var} eigenmodes; (b) illustration of the method due to Yamazaki and Shinozuka [62]; (c) proposed method for simulation of cross correlated fields in two steps when components share the same distribution; (d) proposed method for components with different distributions, where eigenanalysis of each field is performed separately.

identically good results as the proposed scheme depicted in the third part (c) of the figure: decomposition into (i) the preparation of a vector of cross correlated random variables $\boldsymbol{\chi}^{\mathbf{D}}$ and (ii) the expansion of each random field H_i using a subset $\boldsymbol{\chi}_i^{\mathbf{D}}$ and always the same orthogonal base as in (a). The advantage of the proposed procedure (c) is that the simulation of each random field can be done separately using either a KLE or EOLE base while the cross correlated random variables $\boldsymbol{\chi}^{\mathbf{D}}$ are prepared in advance. Incorporation into an existing algorithm for simulation of univariate fields is therefore simple and transparent.

4. Transformation to non-Gaussian random fields

In most applications, the Gaussian random field $\mathbf{H}$ is often used to model uncertainties with spatial variability because of convenience and a lack of alternative models. However, the Gaussian model is not applicable in many situations. For example, it cannot be used to model Young's modulus or the strength of a material, which is always positive.

Let us denote the marginal cumulative [probability] distribution function (cdf) of each component $\widetilde{H}_i$ of the non-Gaussian vector random field $\widetilde{\mathbf{H}}$ by G_i [g_i]. In the discretized version, one can assemble the target correlation matrix $\widetilde{\mathbf{F}}$ of all random fields by computing the entries $\widetilde{F}_{k,l}^{i,j}$ as a product of the autocorrelation coefficient $\widetilde{F}_{\mathbf{uu}}^{i,j}$ (depending only on the positions of each pair of points, Eq. (2)) and the target cross correlation $\widetilde{C}^{i,j}$. It would be convenient to find an underlying Gaussian random field $\mathbf{H}$ (with some cross correlation matrix $\mathbf{C}$ studied earlier) that can be easily transformed into the target field $\widetilde{\mathbf{H}}$ while keeping the target cross correlation matrix between these components denoted by $\widetilde{\mathbf{C}}$. The univariate nonlinear transformation of the Gaussian variables [called the translation process by Grigoriu [19]] is the mapping $h_i(\cdot)$:

$$\widetilde{H}_i(x_k) = h_i[H_i(x_k)] = G_i^{-1}\{\Phi[H_i(x_k)]\}, \quad i = 1,\ldots,N_F;\ k = 1,\ldots,N \tag{29}$$

where $\Phi(\cdot)$ is the standard cumulative Gaussian probability function. A good reason for transforming the variables from the standard Gaussian space into the original non-Gaussian space is that the former is often required in advanced reliability and sensitivity analyses.

The Nataf [33] model has been proposed by Liu and Der Kiureghian [27] for transforming non-Gaussian multivariate distribution into standardized Gaussian distribution. We will show how the Nataf model can be used within the presented framework for effective simulation of cross correlated Gaussian random fields in order to model non-Gaussian fields with prescribed marginal distributions G_i, the autocorrelation function from Eq. (2) and cross correlated via $\mathbf{C}$. For application of the Nataf model, the correlation coefficient $\widetilde{\rho}_{i,j}$ of each pair (i,j) of non-Gaussian variables must be adjusted to form the correlation coefficient $\rho_{i,j}$ of a pair of Gaussian variables. The adjustment has been shown [27] to be a unique solution of the following twofold integral Eq. (12) in [27]:

$$\tilde{\rho}_{i,j} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\widetilde{H}_i - \mu_i}{\sigma_i}\right)\left(\frac{\widetilde{H}_j - \mu_j}{\sigma_j}\right)\varphi(H_i, H_j, \rho_{i,j})\,\mathrm{d}H_i\,\mathrm{d}H_j \tag{30}$$

where the values of the original variables $\widetilde{H}_i$, $\widetilde{H}_j$ (with the means μ_i, μ_j and standard deviations σ_i, σ_j) are expressed in terms of the standard Gaussian variables in the spirit of Eq. (29) via $\widetilde{H}_i = G_i^{-1}[\Phi(H_i)]$. Function $\varphi(H_i, H_j, \rho_{i,j})$ is the standard bivariate Gaussian density.

Due to the uniqueness of the solution, the relationship in Eq. (30) can be expressed as a correction to the original correlation [27]:

$$\rho_{i,j} = \kappa\tilde{\rho}_{i,j} \tag{31}$$

In general, the correction factor κ (satisfying $\kappa \geqslant 1$) is a function of both marginal distributions and their correlation: $\kappa = \kappa(G_i, G_j, \tilde{\rho}_{i,j})$. For some pairs of distributions κ becomes just a constant or a function of only some of the three types of information. Other important properties are that (i) $\rho_{i,j} = 0$ for $\tilde{\rho}_{i,j} = 0$, (ii) $|\tilde{\rho}_{i,j}| \leqslant |\rho_{i,j}|$ and that (iii) $\tilde{\rho}_{i,j}$ is a strictly increasing function of $\rho_{i,j}$.

In application of the Nataf model, we seek the corresponding correlation matrix $\mathbf{F}$ of a Gaussian random vector field $\mathbf{H}$. The correct method is to solve the correction factor for each entry $F^{i,j}_{k,l} = \kappa^{i,j}_{kl}\widetilde{F}^{i,j}_{k,l}$ depending on G_i, G_j and $\widetilde{F}^{i,j}_{k,l}$. Unfortunately, the full-block correlation matrix $\mathbf{F}$ does not follow the simple pattern from Eq. (20) any more. In particular, the diagonal blocks are not identical anymore, because the distributions G_i may differ in general, and the off-diagonal blocks are not a simple multiple of the diagonal block (for the same reasons). Even if the distributions G_i were identical, the corrections κ would prevent each off-diagonal block from being a simple multiple of the diagonal block, because in general $C^{i,j}$s are not zeros (and also generally are not all the same).

We remark also that not every combination of the autocorrelation structure with the non-Gaussian marginal distribution can be admissible for the mapping via underlying Gaussian random field. There are two possible incompatibilities. The first one arises when the autocorrelation functions of the non-Gaussian fields do not have a corresponding admissible correlations in the Gaussian space (this happens often in cases of high negative correlations combined with strongly non-Gaussian marginals). The second incompatibility arises when the autocorrelation function (or matrix) in the Gaussian space becomes non-positive definite and, therefore, not admissible. The second problem can sometimes be remedied by ignoring negative eigenvalues and corresponding eigenmodes.

From the preceding paragraphs, it becomes clear that the presented approach for simulation of Gaussian vector random fields can not generally be employed for simulation of vector random fields with arbitrary marginals. However, it is known that for the majority of commonly used continuous distributions the correction factors κ are only slightly greater than one [27]. Therefore, the difference between correlation matrices $\mathbf{F}$ and $\widetilde{\mathbf{F}}$ is usually very small. The difference actually depends on the "non-Gaussianity" of the distributions G_i. The closer the component distributions G_i are to the elliptical distributions (Gaussian inclusive), the closer these two matrices are. In the following paragraphs we will try to find an approximation $\mathbf{F}'$ of the correct Nataf full correlation matrix $\mathbf{F}$ in order to be able to profit from the presented framework for Gaussian fields.

The six-step procedure from Section 3.3 must be slightly reviewed (see Fig. 1d). The corresponding steps of the two series expansion methods KLE and EOLE follow:

(1) Given the common autocorrelation function in the original (non-Gaussian) space, a Nataf correction function $\kappa_i(\tilde{\rho}) = \kappa(G_i, G_i, \tilde{\rho})$ must be found for each field $i = 1, \ldots, N_F$ over the whole range of autocorrelation coefficients $\tilde{\rho}$. The set of functions κ_i transform the original correlations into the Gaussian space. Then, the spectral analysis of the autocorrelation structure for each underlying Gaussian field and the choice of the common number of eigenmodes N_{var} is made.
   (a) *KLE*: Decomposition of the autocovariance function multiplied by $\kappa_i(\tilde{\rho})$ resulting in eigenvalues and eigenfunctions over the domain $\hat{\Omega}$.
   (b) *EOLE*: Based on the discretization (grid of points) the autocorrelation matrices $\mathbf{F}^i_{\mathbf{uu}}$ are assembled and the corresponding sets of N_{var} largest eigenvalues and the corresponding eigenvectors ($\mathbf{\Phi}^{\mathbf{u}}_i$ and $\mathbf{\Lambda}^{\mathbf{u}}_i$ matrices) are determined.

Note that if all marginals are identical, all autocorrelation functions (matrices) match. By identical marginals we mean that all components must share the same type of distribution and these distributions must have identical shape parameters. For example, two Weibull distributions with an identical coefficient of variation have the same shape parameter m (usually called the Weibull modulus). In most cases, the Nataf's correction will be only slightly greater than one over the whole range of possible autocorrelations and thus the eigenvalues and eigenfunctions [vectors] will be very similar for each Gaussian field. Therefore, one can solve one field only and use iterations to refine the eigenmodes for the other fields.

(2) Find a corrected cross correlation matrix $\mathbf{C}$ given the target matrix $\widetilde{\mathbf{C}}$ and marginals $G_1,\ldots,G_{N_F}$ using the Nataf model. Each off-diagonal entry is obtained as $C^{i,j}=\kappa^{i,j}\widetilde{C}_{i,j}$, $(i,j=1,\ldots,N_F)$. Then, eigenvalues must be computed with corresponding orthogonal eigenpairs $(\mathbf{\Phi}^{\mathbf{C}}_{\mathbf{I}},\mathbf{\Lambda}^{\mathbf{C}}_{\mathbf{I}})$ of the target cross correlation matrix $\mathbf{C}$ among random fields. The choice of number of eigenmodes $N_{F,r}\leqslant N_F$ is made.

(3) Simulation of Gaussian random vector $\boldsymbol{\xi}$ of $N_r=N_{F,r}N_{\mathrm{var}}$ independent standard Gaussian variables ξ_j (see step 3 in Section 3.3 for details).

(4) Simulation of cross correlated random vector $\boldsymbol{\chi}^{\mathbf{D}}$ (see step 4 in Section 3.3 for details).

(5) Simulation of all underlying Gaussian fields $i=1,\ldots,N_F$ one-by-one using the correct portion of $\boldsymbol{\chi}^{\mathbf{D}}$ and eigenmodes from step 1 (see Fig. 1d for illustration). Depending on the method, the fields are composed in analogy with step 5 in Section 3.3.

(6) Translate the underlying Gaussian random fields into non-Gaussian via Eq. (29).

In the procedure, we have made a certain simplification of the consistent approach described above, so it is important to assess the error of the approximation. Assume that the distribution of the underlying Gaussian random field is simulated correctly. Then the non-Gaussian field obtained by the memoryless transformation has no error in the marginal distributions. The only error can arise is in the correlation structure of the fields. Obviously, every field alone has a correct autocorrelation structure, because it is expanded using independent Gaussian variables via orthogonal transformation of correct correlation matrices. Let us now take a look at the cross correlations obtained with the suggested approach.

In the EOLE method, the simulation of the nodal point values of all fields (step 5) can be written simply as:

$$\mathbf{H}=\mathbf{\Phi}^{\mathbf{E}}(\mathbf{\Lambda}^{\mathbf{E}})^{1/2}\boldsymbol{\chi}^{\mathbf{D}} \tag{32}$$

where $\mathbf{\Phi}^{\mathbf{E}}$ and $\mathbf{\Lambda}^{\mathbf{E}}$ are the eigenvector and eigenvalue matrices of a (block-diagonal) correlation matrix $\mathbf{E}$ that is constructed as follows. Matrix $\mathbf{E}$ consists of diagonal blocks $\mathbf{F}^{i,i}_{\mathbf{uu}}$; all off-diagonal blocks are zero matrices. Therefore, the eigenvalue [eigenvector] $\mathbf{\Lambda}^{\mathbf{E}}[\mathbf{\Phi}^{\mathbf{E}}]$ matrices have the matrices $\mathbf{\Lambda}^{\mathbf{u}}_i[\mathbf{\Phi}^{\mathbf{u}}_i]$ on the diagonal blocks and zeros elsewhere. By substituting Eq. (23) into Eq. (32) we obtain the fields in terms of transformation of independent variables $\boldsymbol{\xi}$:

$$\mathbf{H}=\underbrace{\mathbf{\Phi}^{\mathbf{E}}(\mathbf{\Lambda}^{\mathbf{E}})^{1/2}\mathbf{\Phi}^{\mathbf{D}}(\mathbf{\Lambda}^{\mathbf{D}})^{1/2}}_{\mathbf{\Psi}^{\mathbf{F}}}\boldsymbol{\xi}=\mathbf{\Psi}^{\mathbf{F}}\boldsymbol{\xi} \tag{33}$$

Therefore, the resulting full (block) correlation matrix can be computed as $\mathbf{F}'=\mathbf{\Psi}^{\mathbf{F}}[\mathbf{\Psi}^{\mathbf{F}}]^{\mathrm{T}}$:

$$\begin{aligned}\mathbf{F}' &= \mathbf{\Phi}^{\mathbf{E}}\cdot(\mathbf{\Lambda}^{\mathbf{E}})^{1/2}\mathbf{\Phi}^{\mathbf{D}}\cdot(\mathbf{\Lambda}^{\mathbf{D}})^{1/2}\cdot[\mathbf{\Phi}^{\mathbf{E}}\cdot(\mathbf{\Lambda}^{\mathbf{E}})^{1/2}\mathbf{\Phi}^{\mathbf{D}}\cdot(\mathbf{\Lambda}^{\mathbf{D}})^{1/2}]^{\mathrm{T}}\\ &= \mathbf{\Phi}^{\mathbf{E}}\cdot(\mathbf{\Lambda}^{\mathbf{E}})^{1/2}\underbrace{\mathbf{\Phi}^{\mathbf{D}}\cdot(\mathbf{\Lambda}^{\mathbf{D}})^{1/2}\cdot(\mathbf{\Lambda}^{\mathbf{D}})^{1/2}[\mathbf{\Phi}^{\mathbf{D}}]^{\mathrm{T}}}_{\mathbf{D}}\cdot(\mathbf{\Lambda}^{\mathbf{E}})^{1/2}[\mathbf{\Phi}^{\mathbf{E}}]^{\mathrm{T}}=\mathbf{\Phi}^{\mathbf{E}}(\mathbf{\Lambda}^{\mathbf{E}})^{1/2}\cdot\mathbf{D}\cdot(\mathbf{\Lambda}^{\mathbf{E}})^{1/2}[\mathbf{\Phi}^{\mathbf{E}}]^{\mathrm{T}}\end{aligned} \tag{34}$$

By this construction, the $\mathbf{F}'$ matrix can be written in square blocks (each of order N):

$$\mathbf{F}'=\begin{array}{c}\\ \mathbf{H}_1\\ \vdots\\ \mathbf{H}_i\\ \vdots\\ \mathbf{H}_{N_F}\end{array}\begin{array}{c}\begin{array}{cccccc}\mathbf{H}_1 & \ldots & \mathbf{H}_j & \ldots & \mathbf{H}_{N_F}\end{array}\\ \begin{pmatrix}\mathbf{F}'_{1,1} & \ldots & \mathbf{F}'_{1,j} & \ldots & \mathbf{F}'_{1,N_F}\\ \vdots & \ldots & \vdots & \vdots & \vdots\\ \mathbf{F}'_{i,1} & \ldots & \mathbf{F}'_{i,j} & \vdots & \mathbf{F}'_{i,N_F}\\ \vdots & \ldots & \vdots & \ddots & \vdots\\ \mathbf{F}'_{N_F,1} & \ldots & \mathbf{F}'_{N_F,j} & \ldots & \mathbf{F}'_{1,N_F}\end{pmatrix}\end{array} \tag{35}$$

Of course, the matrix is symmetric as a whole, but the blocks are not symmetric in general. Using Eq. (34), each block i,j can be written as

$$\mathbf{F}'_{i,j} = C^{i,j} \underbrace{\boldsymbol{\Phi}_i^{\mathbf{u}} (\boldsymbol{\Lambda}_i^{\mathbf{u}})^{1/2}}_{\boldsymbol{\Psi}_i^{\mathbf{u}}} \underbrace{(\boldsymbol{\Lambda}_j^{\mathbf{u}})^{1/2} [\boldsymbol{\Phi}_j^{\mathbf{u}}]^{\mathrm{T}}}_{[\boldsymbol{\Psi}_j^{\mathbf{u}}]^{\mathrm{T}}} \tag{36}$$

The $\mathbf{F}'$ matrix (consisting of blocks $\mathbf{F}'_{i,j} = C^{i,j} \boldsymbol{\Psi}_i^{\mathbf{u}} [\boldsymbol{\Psi}_j^{\mathbf{u}}]^{\mathrm{T}}$) represents a good approximation of $\mathbf{F}$ in most cases (see Section 5.1 for a numerical example with an estimation of error). The diagonal blocks are equal to the autocorrelation of each field $\mathbf{F}'_{i,i} = \mathbf{F}^i_{\mathbf{uu}}$. The off-diagonal blocks $\mathbf{F}'_{i,j}$, in a certain sense, inherit a combination of the autocorrelations of the ith and jth random field (a product of the eigenmodes of both). Note that if a pair of fields i, j follow an identical autocorrelation structure, the corresponding cross correlation block is just a $C^{i,j}$ multiple of it (compare to Eq. (20)).

The $\mathbf{F}'$ can be computed and compared to $\mathbf{F}$ before performing any simulations. If the difference (cross correlation errors) is not acceptable for the analyst and he wants to return to the consistent procedure employing the correct Nataf transformation for $\mathbf{F}$ in the orthogonal transformation via Eq. (28), we recommend to use Eq. (33) to find a very good approximation of the eigenmodes of $\mathbf{F}$ needed in Eq. (28). The eigenmodes can be refined iteratively.

5. A numerical example

An example of an application is a model of the random spatially varying material properties of concrete structure. In the first version of this, we assume that the random fields are Gaussian (extension to non-Gaussian fields follows). Such an example may have direct application in stochastic nonlinear modeling of random resistance in quasibrittle structures, i.e. structures manifesting a combined deterministic-statistical size effect on strength see e.g. [3]. The suggested method is compared to the more general framework where the multiplicative properties of correlation matrices are not used. A simple error assessment of simulated samples follows in Section 6 and is used to demonstrate the accuracy of the technique.

For the model a simply supported four-point bending beam has been chosen. The beam is modeled in two dimensions; the effect of beam width is ignored. Three parameters of the nonlinear constitutive law of concrete are considered random, namely the local tensile strength f_t, local E-modulus and local fracture energy G_F. These parameters are firstly modeled by a three-variate stationary isotropic standard Gaussian random field with a common Gaussian autocorrelation function (Eq. (2), $pow = \dim = 2$) and a simple cross correlation matrix of order $N_F = 3$. Transformation to a non-Gaussian distribution is studied in the following subsection.

The region of interest represented by the beam is a rectangle 0.4 times 0.1 m. The autocorrelation length $d_i = 0.05$ m, $i = 1,2$. To avoid possible disturbances at boundaries, we artificially enlarge the discretization region with one autocorrelation along all sides, so Ω is a rectangle 0.5 by 0.2 m.

The discretization method chosen is the EOLE method. The optimal grid is represented by $N = 1000$ points, a regular lattice of 50 points (x-direction) by 20 points (y-direction). Following the scheme in Section 3.3, the square symmetric and positive definite autocorrelation matrix $\mathbf{F}_{\mathbf{uu}}$ of order N is assembled and the N real positive eigenvalues are computed. The package EISPACK has been used for this purpose and the total time spent on computation was on average 7.8 seconds. It is only necessary to compute 114 eigenmodes with the largest eigenvalues at a given acceptable variability of 99 %. The value of error measure ε_{Fu} based on a portion of the largest eigenvalues considered is plotted in Fig. 2a. Instead of $N = 1000$ we consider $N_{\mathrm{var}} = 114$ eigenmodes only. The resulting eigenvalue and eigenvector matrices of order N_{var} are denoted $\boldsymbol{\Lambda}^{\mathbf{u}}$ and $\boldsymbol{\Phi}^{\mathbf{u}}$ respectively.

The second step in the proposed procedure is the computation of eigenvectors of a given cross correlation matrix $\mathbf{C}$. In our case the matrix represents the three material properties and the correlations have been chosen as follows:

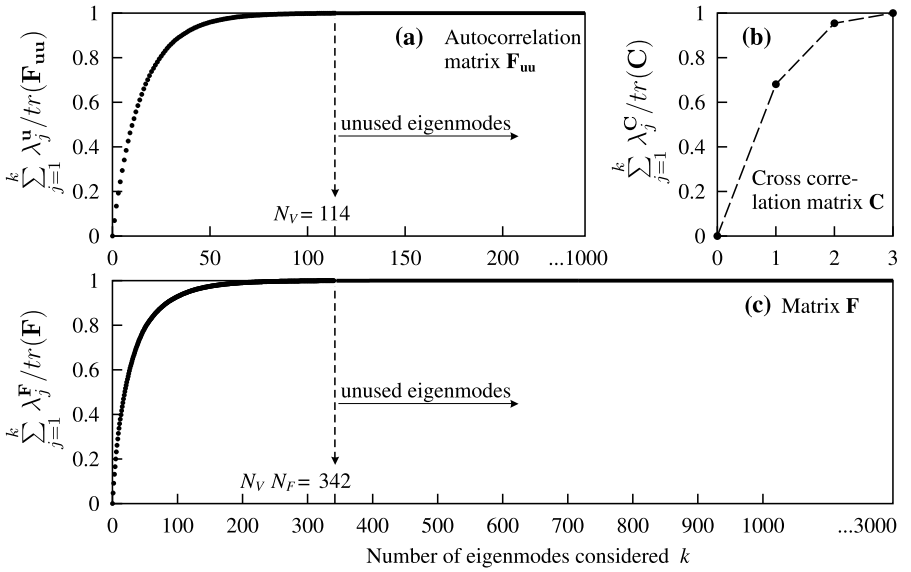

Fig. 2. Reduction of number of eigenmodes of correlation matrices based on their contribution to the trace of eigenvalue matrix. Eigenvalues are sorted such that $\lambda_i \geqslant \lambda_j$ for $i < j$.

$$\mathbf{C} = \begin{array}{c} \\ f^t \\ E \\ G_F \end{array} \begin{array}{c} \begin{array}{ccc} f^t & E & G_F \end{array} \\ \begin{pmatrix} 1 & 0.8 & 0.2 \\ & 1 & 0.5 \\ sym. & & 1 \end{pmatrix} \end{array} \tag{37}$$

The eigenvectors and corresponding eigenvalues are:

$$\mathbf{\Phi^C} = \begin{array}{c} \begin{array}{ccc} \mathbf{\Phi}_1^{\mathbf{C}} & \mathbf{\Phi}_2^{\mathbf{C}} & \mathbf{\Phi}_3^{\mathbf{C}} \end{array} \\ \begin{pmatrix} 0.599 & -0.511 & 0.616 \\ 0.671 & -9.96\ 10^{-2} & -0.735 \\ 0.437 & 0.854 & 0.283 \end{pmatrix} \end{array} \quad \text{and} \quad \mathbf{\Lambda^C} = diag \begin{array}{c} \begin{array}{ccc} \mathbf{\Phi}_1^{\mathbf{C}} & \mathbf{\Phi}_2^{\mathbf{C}} & \mathbf{\Phi}_3^{\mathbf{C}} \end{array} \\ \begin{pmatrix} 2.041 & 0.822 & 0.137 \end{pmatrix} \end{array} \tag{38}$$

In this case, no reduction according to Eq. (13) of the number of eigenmodes taken into account can be done without a significant loss of accuracy (see Fig. 2b) and therefore $N_{F,r} = N_F = 3$.

Eqs. (21), (22) could now be used to determine the eigenmodes of $\mathbf{F}$ and to avoid solving the full eigenvalue problem needed in the method of Yamazaki and Shinozuka [62]. In terms of accuracy, there is no difference between the two methods from Fig. 1b and c is used. The only difference is that the computation of eigenvectors and eigenvalues ($\mathbf{\Phi^F}$ and $\mathbf{\Lambda^F}$) would take much more computer time and memory if the information about them (Eqs. (21), (22)) were not used. In our numerical example the solution of the eigenmodes of matrix $\mathbf{F}$ takes approximately 230 s which is 30 times more compared to the case when two separate eigenproblems are solved ($\mathbf{C}$ and $\mathbf{F_{uu}}$). In general, the savings achieved by the proposed method depend on whether the simulated random fields share an identical autocorrelation structure (of the underlying Gaussian field) or not. In the discrete case (EOLE), the proposed method requires to solve once an eigenproblem of dimension N_F plus one (or maximum N_F times similar) separate eigenproblem(s) of size N. This will always be faster then solving the eigenproblem of size $N \times N_F$. In terms of the memory requirements for the eigenvectors, the proposed method needs to save $N^2 + N_F^2$ numbers (or maximum $N_F \times N^2 + N_F^2$ numbers in the case of N_F different marginals) while the full approach requires to save $N^2 \times N_F^2$ numbers. This is a noticeable difference.

The next step is the simulation of random vector $\boldsymbol{\xi}_r$ of $N_r = N_{F,r} \times N_{\text{var}}$ independent Gaussian variables. The result of this step is an $N_r \times N_{\text{sim}}$ random matrix. Therefore we must simulate $N_{\text{var}} \times N_F = 114 \times 3 = 342$ independent random variables each represented by N_{sim} realizations. For this purpose we used two alternatives, namely crude Monte Carlo sampling (MC) and Latin Hypercube Sampling (LHS).

For the simulation of cross correlated random vector $\boldsymbol{\chi}^{\mathbf{D}}$ (step 4) by matrix multiplication (Eq. (23) or Eq. (24)) the eigenvector and eigenvalue matrices $\boldsymbol{\Phi}^{\mathbf{D}}$ and $\boldsymbol{\Lambda}^{\mathbf{D}}$ are needed. They are obtained simply by Eqs. (14) and (15) where the matrix from step (2) and a random matrix from step (3) are utilized. In our case $\boldsymbol{\Phi}^{\mathbf{D}}$ reads:

$$\boldsymbol{\Phi}^{\mathbf{D}} = \begin{pmatrix} \overbrace{\begin{pmatrix} 0.599 & 0 & \cdots & 0 \\ 0 & 0.599 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0.599 \end{pmatrix}}^{\boldsymbol{\Phi}^{\mathbf{D}}_1} & \cdots & \overbrace{\begin{pmatrix} 0.616 & 0 & \cdots & 0 \\ 0 & 0.616 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0.616 \end{pmatrix}}^{\boldsymbol{\Phi}^{\mathbf{D}}_{N_F}} \\ \begin{pmatrix} 0.671 & 0 & \cdots & 0 \\ 0 & 0.671 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0.671 \end{pmatrix} & \ddots & \vdots \\ \begin{pmatrix} 0.437 & 0 & \cdots & 0 \\ 0 & 0.437 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0.437 \end{pmatrix} & \cdots & \ddots \end{pmatrix} \tag{39}$$

The last step is the simulation of all random fields $i = 1, 2, \ldots, N_F$ by the EOLE method introduced in Eq. (7) for each random field, i.e. by using Eq. (27). Since we do not need the continuous representation of fields the EOLE interpolation can be skipped and we can obtain the discrete grid values of each ith field in the spirit of Karhunen–Loève expansion simply by the orthogonal transformation:

$$\widehat{\mathbf{H}}_i = \sum_{j=1}^{N_{\text{var}}} \sqrt{\lambda_j^{\mathbf{u}}} \chi_{i,j}^{\mathbf{D}} [\boldsymbol{\Phi}_j^{\mathbf{u}}]^{\mathrm{T}} \tag{40}$$

One randomly chosen realization of the three fields is plotted in Fig. 3a,b and c. In the same figure it can be seen how the cross correlation of fields influences the shape similarity of corresponding realizations. Fig. 3d illustrates the typical plot of the mean and variance profiles of the field over the target domain Ω. Such a plot serves as visual check for the accuracy of simulations of fields and is commented on in the following Section 6.

5.1. Transformation to a non-Gaussian random field

As mentioned earlier, the Gaussian model may not always be applicable. Let us now show the impact of non-Gaussianity for the errors introduced by the present method. Let us now consider the tensile strength f_{t} and fracture energy G_{f} Weibull distributed, and the modulus of elasticity E lognormal distributed with the mean values and standard deviations given in parentheses: $G_{f_{\text{t}}} \sim W(4, \underset{\sim}{1})$, $G_{G_f} \sim W(100, 15)$ and $G_E \sim L(40, 4)$ (cov = 25%, 15% and 10%). The target cross correlation matrix $\tilde{\mathbf{C}}$ is identical to that in Eq. (37). The corresponding correlations between the three variables in the Gaussian space according to the Nataf model read:

$$\mathbf{C} = \begin{array}{c} \\ f_t \\ E \\ G_F \end{array} \begin{array}{c} \begin{array}{ccc} f_t & E & G_F \end{array} \\ \begin{pmatrix} 1 & 0.8053 & 0.2017 \\ \vdots & 1 & 0.5076 \\ \text{sym.} & \cdots & 1 \end{pmatrix} \end{array} \tag{41}$$

As can be seen, the correlations are only slightly increased in the Gaussian space. Also the eigenvalues $\boldsymbol{\Lambda}^{\mathbf{C}} = \text{diag}(2.051, 0.820, 0.130)$ and the eigenvectors:

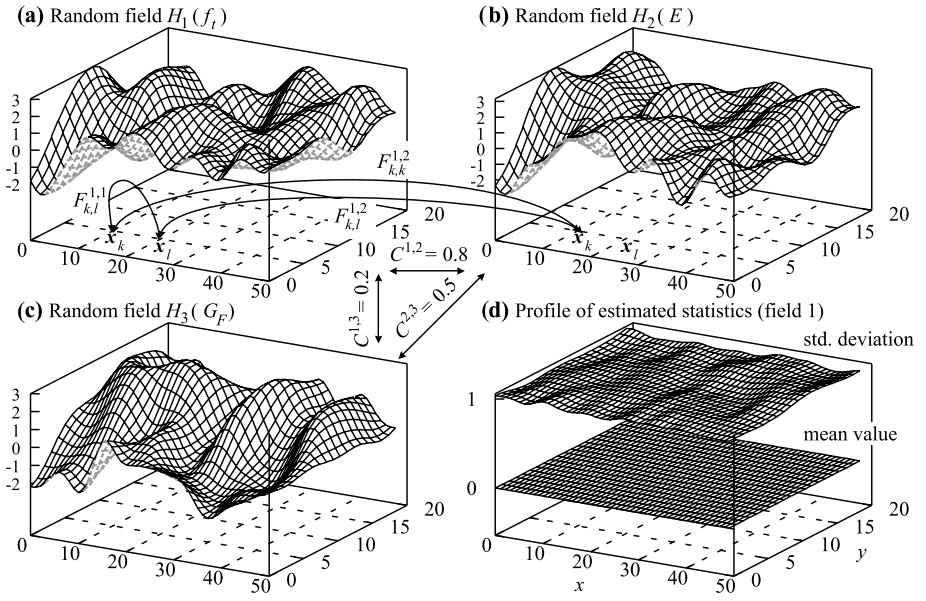

Fig. 3. (a)–(c) Random realization of simulated three-variate Gaussian random field, illustration of the meaning of correlation coefficients $F^{i,j}_{k,l}$. (d) Profile of the estimated mean value and standard deviation of f_t-field ($N_{\text{sim}} = 1000$).

$$\mathbf{\Phi}^{\mathbf{C}} = \begin{array}{c} \\ \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{\Phi}^{\mathbf{C}}_1 & \mathbf{\Phi}^{\mathbf{C}}_2 & \mathbf{\Phi}^{\mathbf{C}}_3 \end{array} \\ \begin{pmatrix} 0.598 & -0.515 & 0.614 \\ 0.671 & -9.8\ 10^{-2} & -0.735 \\ 0.439 & 0.852 & 0.286 \end{pmatrix} \end{array} \tag{42}$$

are nearly identical to those in Eqs. (38) and the difference cannot be seen in a common plot (Fig. 2b).

To show the errors due to simplification proposed in this paper, we select only three discretization points from the domain Ω. Their autocorrelation matrix is again assembled via Eq. ((2), $pow = \dim = 2$). The points are selected so that the correlations $\widetilde{\mathbf{F}}_{\mathbf{uu}}$ (see Eq. (43)) cover most of the range (0;1). The three non-Gaussian components of the vector random field must have three different autocorrelation matrices of the corresponding Gaussian random fields, which are obtained by inverse solution of Eq. (30) for each entry of $\widetilde{\mathbf{F}}_{\mathbf{uu}}$. In the standard Gaussian space, the three autocorrelation matrices are positive definite and read:

$$\widetilde{\mathbf{F}}_{\mathbf{uu}} = \begin{array}{c} \\ \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \end{array} \begin{array}{c} \begin{array}{ccc} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{array} \\ \begin{pmatrix} 1 & 0.9 & 0.1 \\ \vdots & 1 & 0.5 \\ \text{sym.} & \cdots & 1 \end{pmatrix} \end{array}, \qquad \mathbf{F}^{(1)}_{\mathbf{uu}} = \begin{pmatrix} 1 & 0.90034 & 0.10027 \\ \vdots & 1 & 0.50083 \\ \text{sym.} & \cdots & 1 \end{pmatrix}$$

$$\mathbf{F}^{(2)}_{\mathbf{uu}} = \begin{pmatrix} 1 & 0.90045 & 0.10045 \\ \vdots & 1 & 0.50124 \\ \text{sym.} & \cdots & 1 \end{pmatrix}, \qquad \mathbf{F}^{(3)}_{\mathbf{uu}} = \begin{pmatrix} 1 & 0.90145 & 0.10145 \\ \vdots & 1 & 0.504 \\ \text{sym.} & \cdots & 1 \end{pmatrix} \tag{43}$$

The full correlation matrix $\mathbf{F}'$ in the Gaussian space computed via Eqs. (35) and (36) is displayed in the upper triangle of the following matrix in Eq. (44). We use the symmetry of the matrix to save space and display the "correct" correlations $\mathbf{F}$ obtained by applying the Nataf transformation to all entries in the lower triangle. By comparing the upper and lower triangles, one can see that they do not match. This indicates that there are errors due to our simplification. However, the correlation errors in the Gaussian space are negligibly small in our case. Let us focus on the diagonal blocks. It is a direct implication of the selected strategy of employing a set of independent variables to represent each field that the autocorrela-

tion structure of each field is fulfilled exactly (see Fig. 1d). Regarding the off-diagonal blocks of $\mathbf{F}'$: the diagonal entries seem to be exact (i.e. equal to those of the $\mathbf{F}$ matrix), but in fact they are not. The errors in these terms are few orders of magnitude smaller than those in the off-diagonal entries, but still, the diagonal terms are not equal to $C^{i,j}$, because the eigenvectors in Eq. (36) are not orthogonal. It might be interesting to quantify how large the correlation error is. Let us define the mean square ρ_{rms} correlation error as the square root of the sum of squared differences between $\mathbf{F}$ and $\mathbf{F}'$ counted in the upper triangle. In the present example with three fields and three discretization points ρ_{rms} equals 0.006. The error equals 0.03 for the case when we completely ignore the Nataf's corrections of $\widetilde{\mathbf{C}}$ and autocorrelations $\widetilde{\mathbf{F}}_{\mathbf{uu}}$ and simply use the original correlation matrices in the scheme (i.e., $\mathbf{F}'$ would equal $\widetilde{\mathbf{F}}$). Note that the blocks of the exact $\mathbf{F}$ matrix are symmetric while the blocks of the $\mathbf{F}'$ are not. Let us also note that the "correct" $\mathbf{F}$ matrix is not positive definite and therefore it has some negative eigenvalues. This fact would complicate the approach from Fig. 1b. The approximate matrix $\mathbf{F}'$ is positive definite (this matrix would in fact be estimated using a large simulated sample of the fields).

$$
\begin{array}{c}
\\
\mathbf{H}_1 \\
\mathbf{H}_2 \\
\mathbf{H}_3
\end{array}
\begin{array}{c}
\begin{array}{ccc} \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_3 \end{array} \\
\begin{pmatrix}
\left(\mathbf{F}^{(1)}_{\mathbf{uu}}\right) & \begin{pmatrix} 0.8053 & 0.7251 & 0.0811 \\ 0.7250 & 0.8053 & 0.4036 \\ 0.0806 & 0.4033 & 0.8053 \end{pmatrix} & \begin{pmatrix} 0.2017 & 0.1818 & 0.0208 \\ 0.1816 & 0.2017 & 0.1016 \\ 0.0199 & 0.1011 & 0.2017 \end{pmatrix} \\
\begin{pmatrix} C^{1,2} & 0.7246 & 0.0803 \\ \vdots & \ddots & 0.4021 \\ \text{sym.} & \cdots & C^{1,2} \end{pmatrix} & \left(\mathbf{F}^{(2)}_{\mathbf{uu}}\right) & \begin{pmatrix} 0.5076 & 0.4575 & 0.0522 \\ 0.4571 & 0.5076 & 0.2556 \\ 0.0503 & 0.2546 & 0.5076 \end{pmatrix} \\
\begin{pmatrix} C^{1,3} & 0.1815 & 0.0202 \\ \vdots & \ddots & 0.1009 \\ \text{sym.} & \cdots & C^{1,3} \end{pmatrix} & \begin{pmatrix} C^{2,3} & 0.4566 & 0.0506 \\ \vdots & \ddots & 0.2532 \\ \text{sym.} & \cdots & C^{2,3} \end{pmatrix} & \left(\mathbf{F}^{(3)}_{\mathbf{uu}}\right)
\end{pmatrix}
\end{array}
\qquad
\begin{pmatrix} \diagdown\, \mathbf{F}' \\ \diagup\, \mathbf{F} \end{pmatrix} = \tag{44}
$$

The correlation errors also propagate into the original space of non-Gaussian components. The prediction of these errors is straightforward: all elements of the $\mathbf{F}'$ matrix must be used in Eq. (30) to transform them into $\widetilde{\mathbf{F}}'$. By comparing such a matrix to $\widetilde{\mathbf{F}}$ one can assess the error of the method without performing any simulation. This can be helpful in making a decision whether to use the proposed approach or whether to employ the (computationally more demanding) approach from Fig. 1b.

$$
\begin{array}{c}
\\
\widetilde{\mathbf{H}}_1 \\
\widetilde{\mathbf{H}}_2 \\
\widetilde{\mathbf{H}}_3
\end{array}
\begin{array}{c}
\begin{array}{ccc} \widetilde{\mathbf{H}}_1 & \widetilde{\mathbf{H}}_2 & \widetilde{\mathbf{H}}_3 \end{array} \\
\begin{pmatrix}
\left(\widetilde{\mathbf{F}}_{\mathbf{uu}}\right) & \begin{pmatrix} 0.8 & 0.7206 & 0.0807 \\ 0.7205 & 0.8 & 0.4015 \\ 0.0802 & 0.4012 & 0.8 \end{pmatrix} & \begin{pmatrix} 0.2 & 0.1802 & 0.0208 \\ 0.1801 & 0.2 & 0.1007 \\ 0.0197 & 0.1002 & 0.2 \end{pmatrix} \\
\begin{pmatrix} 0.8 & 0.72 & 0.08 \\ \vdots & \ddots & 0.4 \\ \text{sym.} & \cdots & 0.8 \end{pmatrix} & \left(\widetilde{\mathbf{F}}_{\mathbf{uu}}\right) & \begin{pmatrix} 0.5 & 0.4508 & 0.0516 \\ 0.4505 & 0.5 & 0.2524 \\ 0.0497 & 0.2514 & 0.5 \end{pmatrix} \\
\begin{pmatrix} 0.2 & 0.18 & 0.02 \\ \vdots & \ddots & 0.1 \\ \text{sym.} & \cdots & 0.2 \end{pmatrix} & \begin{pmatrix} 0.5 & 0.45 & 0.05 \\ \vdots & \ddots & 0.25 \\ \text{sym.} & \cdots & 0.5 \end{pmatrix} & \left(\widetilde{\mathbf{F}}_{\mathbf{uu}}\right)
\end{pmatrix}
\end{array}
\qquad
\begin{pmatrix} \diagdown\, \widetilde{\mathbf{F}}' \\ \diagup\, \widetilde{\mathbf{F}} \end{pmatrix} = \tag{45}
$$

As mentioned earlier, errors only occur in the off-diagonal blocks. In these blocks, the error in the diagonal entries is very small. Note that in practical situations, the correlation error due to the method will be overtaken by the numerical error in the simulation, because the $\mathbf{F}$ or $\mathbf{F}'$ matrices can only result by estimation using a very large number of simulations. This issue will, alongside others, be discussed next.

6. Error assessment of random field simulation

When any method of random field simulation is used, it is required that the statistical characteristics of the generated field be as close to the target parameters as possible. However, the degree of accuracy of the resulting ensemble statistics (such as the mean values, standard deviations, autocorrelation coefficients) is,

in general, satisfactorily high only when the sample size N_{sim} is sufficiently large. The statistical fluctuations in the statistics that arise from generating a finite number of sample functions of the stochastic process or field involved significantly influence the resulting response statistics of the studied model.

Generally, the mean values, the variances, the autocorrelation and spectral density, the cross correlation and the cross spectral density (statistics) cannot be generated with absolute accuracy. Basic information about a random field is captured by its second-moment characteristics, i.e. the estimated mean function and the covariance function. Therefore, each of these statistics can be considered as a random variable. The assessment can be done numerically by performing N_{run} runs of the same simulation process with a different random setting of the seed of the pseudo-random number generator [37,57]. Naturally the statistics obtained in each run are different. In our case the accuracy will be influenced by:

- Number of random variables N_{var} (=number of eigenmodes of a target autocorrelation structure) used for representation of each random field.
- Number of eigenmodes of $\mathbf{C}$ considered.
- Sampling technique used for uncorrelated Gaussian random variables in connection with fulfillment of the correlation structure (independence).
- Number of N_{sim} simulations used (sample size).

Let us first comment on aspects of the simulation of univariate random fields. Several papers exist focusing on the impact of the truncation of the number of eigenmodes N_{var} on the desired statistics of a field for both KLE and EOLE method e.g. [64]. The number of terms in the expansion of a field always introduces an error (e.g. underestimation of the variability of a field, the difference of the actual autocorrelation structure from the target one, etc.). We will however, assume that the accuracy of the deterministic part of the random field simulation (eigenvalues $\mathbf{\Lambda}$ with eigen-functions[vectors] $\psi(\mathbf{x})[\mathbf{\Phi}]$) is given (in the case of non-Gaussian fields, we ignore the difference between $\mathbf{F}$ and $\mathbf{F}'$). Then let us study the influence of the random part on the statistics of random fields. The randomness is introduced by a pseudo-random generator (digital simulation on computers). Therefore, when the seed of the pseudo-random number generator is changed, another random field's sample is generated and naturally, other values of all sample statistics are obtained. The influence on the mean and variance profile of the simulated field and the influence on the correlation structure of a field can be studied separately.

Mean and variance profile: Generally in the case of the Monte Carlo type simulation of random variables the statistics converge to target values for $N_{sim} \to \infty$ and $N_{run} \to \infty$. It can be shown [57] that if LHS is used for representation of random variables, the simulated mean values of a field are exact in discretization points, see the profile in Fig. 3d. The reason for this is that the orthogonal transformation preserves the mean values of the random variables. Averages of those random variables sampled by LHS match the desired mean values exactly (see Eq. (25)).

Regarding the variance profile we note firstly that a general feature of LHS is that higher statistical moments are not represented exactly by samples; the higher the moment the less the estimated moments match the desired ones. This is documented by the slightly uneven profile of estimated standard deviation in Fig. 3d. A comparison of two different techniques of sample simulation in the LHS and its influence on the variance profile of a field is given in [57], and generally it can be said that the improvement of LHS represented by Eq. (25) results in an improvement of the variance profile in comparison with the standard LHS where samples represent medians of each strata.

Reduction of spurious correlation: There are two possibilities for the simulation of random variables. First, the realizations can be simulated at random without any attention paid to the correlations among them. If a good random number generator is used it can be expected that a spurious correlation among simulated random variables converges to zero with increasing N_{sim}. Second, the realizations can be permuted or transformed into non-correlated form before they are used for random field expansion.

In order to demonstrate the influence of the spurious correlation among representing random variables, the LHS methodology has been chosen. It is important to emphasize that in the case of LHS the sample values are not random (using a given algorithm of stratified sampling), *only the sample ordering is random*. In the context of LHS, let us mention the original work of Iman and Conover [25], who described their algorithm based on

the Spearman rank correlation coefficient estimator and on the Cholesky decomposition of the correlation matrix. They also keep the sample values and vary only the sample ordering for each variable, so that each random variable is still represented by the same set of values. In our case however, a more efficient technique for correlation control (to diminish spurious correlations among random variables) has been used [56,58,60]; the method is denoted as ILHS from here on.

Let us assume that a given realization $\boldsymbol{\xi}(\theta_0)$ of a random vector $\boldsymbol{\xi}$ is an $N_{\text{var}} \times N_{\text{sim}}$ matrix. One can calculate its correlation matrix $\mathbf{S}(\theta_0)$ of order N_{var}. For a very low number of simulations the correlation matrix $\mathbf{S}$ cannot be a unit matrix because it cannot be reached by any possible combination of the given sample values. The precision at this level can be measured using a matrix norm, e.g. $\varepsilon_S = \max|S_{k,l}|$, $k \neq l$ or $\varepsilon_S^2 = ((\sum_k \sum_l S_{k,l}^2) - N_{\text{var}})/2$, where $k,l = 1,\ldots,N_{\text{var}}$ for both cases [56].

Each random variable ξ_k ($k = 1,\ldots,N_{\text{var}}$) needed is represented by N_{sim} realizations. In the case when no attention is paid to spurious correlation, all $(N_{\text{sim}}!)^{(N_{\text{var}}-1)}$ possible correlation matrices $\mathbf{S}$ of $\boldsymbol{\xi}(\theta)$ are equally probable. However, the best results are obtained only for those θ's for which the norms ε_S are small. The same will hold for the cross correlation fields, with the difference that the number of random variables will increase from N_{var} to $N_r = N_{F,r} N_{\text{var}}$.

Vořechovský and Novák [57,59] have shown that if LHS methodology is used, the quality of the autocorrelation structure is solely influenced by the distance of the correlation matrix $\mathbf{S}(\theta_0)$ of the simulated random vector $\boldsymbol{\xi}(\theta_0)$ from the unit matrix. Moreover, the matrix norm ε_S has proved itself to be a suitable estimator of the accuracy of the autocorrelation structure already at the level of the simulation of random vector $\boldsymbol{\xi}(\theta_0)$. Here, a study has been made on the influence of such spurious correlation on the quality of the autocorrelation structure of the first random field (f_t). The LHS method has been used for the simulation of random variables, with two numbers of simulations $N_{\text{sim}} = 100$ and 1000. In Fig. 4, there is a comparison of the case where no attention is paid to random spurious correlation (LHS) with the case where spurious correlation is diminished (ILHS) using methodology (a genetic algorithm) from Vořechovský and Novák [56,58]. In the case of there being only a very small number of simulations (with respect to the number of random variables N_r used for representation of a vector random field), $\mathbf{S}$ cannot be a unit matrix and therefore the numerically estimated average autocorrelation function does not match the target one (also the scatterband around the average curve is large). The explanation is clear; a small number N_{sim} leads to a large difference between the actual correlation matrix $\mathbf{S}$ and the unit matrix. A clear indication of this limitation is the fulfillment of the defined norms of correlation matrix $\mathbf{S}$ which can be used as objective functions in an algorithm for diminishing spurious correlation. It also corresponds to a measure $\boldsymbol{\varepsilon}_{\mathbf{F}}$ introduced in the following paragraphs. As the number N_{sim} increases, the simulated autocorrelation structure of a field improves. Simply, if the number of simulations is higher than the number of random variables needed (N_r) the algorithm for correlation control over LHS is

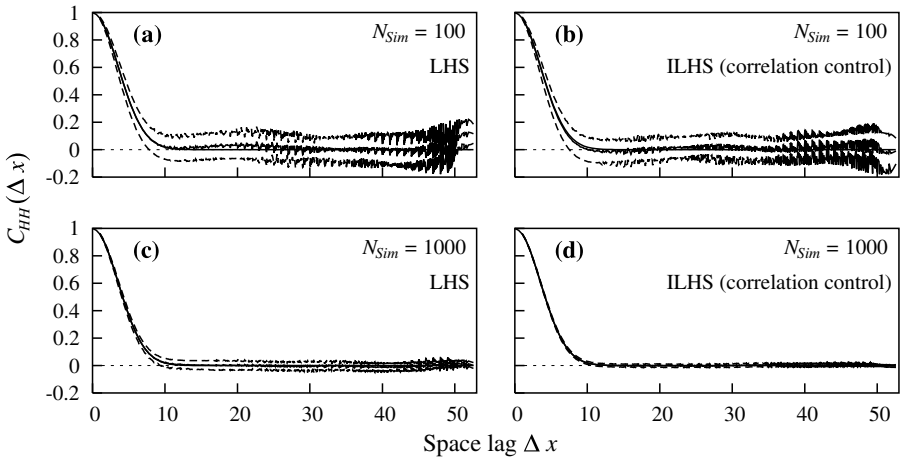

Fig. 4. Comparison of estimated autocorrelation function $C_{\widehat{HH}}(\Delta x)$ for 100 and 1000 simulations (estimated mean value ± standard deviation). Left column: normal LHS. Right column: ILHS (LHS with correlation control).

able to rearrange rank numbers of samples so that the matrix $\boldsymbol{\xi}_r$ variables really represent a set of uncorrelated random variables. Note that the alternative with $N_{\text{sim}} = 1000$ and with the diminished spurious correlation (ILHS) results in an excellent autocorrelation function with almost no variability. This fact corresponds with norm ε_S which was in case (d) very small. It can be seen that the spurious correlation at the level of the simulation of uncorrelated random numbers negatively influences the resulting autocorrelation function of a field. If the variables do not enjoy the property of orthogonality, it simply can not be expected that they will form a precise autocorrelation structure of a field after orthogonal transformation. We conclude that attention should be paid to both the statistics of a random vector (source random variables) and its correlation structure (orthogonality).

In the following paragraphs, we will focus on error assessment of the cross correlation structure among random fields. In order to quantify the error, let the field be discretized into N points (suitably selected grid $\mathbf{u}$). Then, the full-block correlation matrix $\mathbf{F}$ can be simply assembled over the grid. For a given outcome θ_0 of values $\tilde{H}_{\mathbf{u}}(\theta)$ its realization $\hat{\mathbf{F}}$ using a point correlation estimator can also be calculated (for Gaussian fields the Pearson estimator, or generally e.g. Spearman). The quality of the correlation structure of simulated fields can be assessed using the proposed symmetric error matrix $\boldsymbol{\varepsilon}_{\mathbf{F}}$ (Eq. (46)). This matrix contains error terms $\varepsilon_{i,j}$ describing the error of both autocorrelation (diagonal terms) and cross correlation (off-diagonal terms):

$$\boldsymbol{\varepsilon}_{\mathbf{F}} = \begin{array}{c} \\ 1 \\ \vdots \\ i \\ \vdots \\ N_F \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & \dots & j & \dots & N_F \end{array} \\ \begin{pmatrix} \varepsilon_{1,1} & \dots & \varepsilon_{1,j} & \dots & \varepsilon_{1,N_F} \\ & \ddots & \vdots & \ddots & \vdots \\ & & \varepsilon_{i,j} & \dots & \varepsilon_{i,N_F} \\ & sym. & & \ddots & \vdots \\ & & & & \varepsilon_{N_F,N_F} \end{pmatrix} \end{array} \tag{46}$$

Each element i,j may be a matrix norm defined as (i) the maximum difference of correlation coefficients between the target matrix $\mathbf{F}$ and an estimated matrix $\hat{\mathbf{F}}$ of a given outcome θ_0 (largest off-diagonal error):

$$\varepsilon_{i,j} = \max_{1 \leqslant k < l \leqslant N} |F^{i,j}_{k,l} - \hat{F}^{i,j}_{k,l}| \tag{47}$$

which is a conservative measure (row norm), or as (ii) a norm which takes into account deviations of all correlation coefficients:

$$\varepsilon^2_{i,j} = \frac{2}{N(N-1)} \sum_{k=1}^{N_{\text{var}}-1} \sum_{l=k+1}^{N_{\text{var}}} (F^{i,j}_{k,l} - \widehat{F}^{i,j}_{k,l})^2 \tag{48}$$

This estimator $\varepsilon_{i,j}$ is normalized by the total number of entries in the block corresponding to a pair of ith and jth fields and represents the root mean square error in the correlation for each discretization point (k,l). The meaning of the terms is illustrated in Fig. 3.

Elements on the diagonal $\varepsilon_{i,i}$ in the error matrix $\boldsymbol{\varepsilon}_{\mathbf{F}}$ describe the error of the autocorrelation structure of a field H_i while the off-diagonal elements $\varepsilon_{i,j}$ describe the cross correlation error across two fields H_i and H_j. The correlation is estimated using N_{sim} realizations in N discretization points.

It should be noted that the quality of both estimated autocorrelation and cross correlation (difference between $\mathbf{F}$ and $\widehat{\mathbf{F}}$) depends on how far the actual correlation matrix $\widehat{\mathbf{D}}$ of random vector $\boldsymbol{\chi}^{\mathbf{D}}_r$ is from the unity matrix (provided the two sets of eigenmodes are given). Such a conclusion has already been made in the case of a univariate random field [57,59].

7. Applications in modeling of material properties

The presented example is oriented towards resistance modeling through the random material properties of quasibrittle materials. A major phenomenon that is observed in quasibrittle materials is the strong dependence of the peak load on a characteristic dimension of the considered specimen. This phenomenon is commonly referred to as the size effect. A part of the phenomenon has been successfully explained on a

purely energetic basis; it is believed that there exists a purely deterministic size effect caused by energy release associated with stress redistribution prior to failure, see e.g. Bažant and Planas [2]. This part of the size effect dominates in structures of the type known as quasibrittle and plays an important role in small specimen sizes. The deterministic part has already been successfully modeled by nonlinear fracture-mechanics FEM-programs e.g. [38]. There is also a statistical part of size effect driven mainly by extreme value statistics. The statistical (or probabilistic) part of the phenomenon can not be captured by deterministic computations see e.g. [35]. An application of the proposed simulation framework has been recently used for numerical simulation of the Malpasset dam failure [3]. In the example, spatial variability of local material parameters is described by random fields. Each random property is described by its probability density function (PDF) and the rate of fluctuation is governed by the autocorrelation function/length. The correlation length (representing a material property) remains constant for scaled structures, which coincides more or less with reality. The numerical example supports a newly formulated size effect law for crack initiation problems featuring both deterministic and statistical size effects (and the corresponding scaling lengths) and their interaction. A similar kind of study [53] has been performed with dog-bone specimens made of concrete and experimentally tested in uniaxial tension. The specimens were geometrically scaled in two dimensions and the ratio between the largest and smallest size was 1:32. Weibullian random field describing local material strength was used to explain an unusual shape of the size effect curve (dependence of nominal strength on structural size).

Another application of the proposed scheme has recently been used to model the spatially varying strength of parallel filament yarns represented by the fiber bundle model [6,54]. Each random field represents the local random strength of a fiber, but fibers are independent and therefore the corresponding fields have a unit matrix $\mathbf{C}$. It has been shown that the classical Weibull weakest link model (based on the extreme value theory of independent identically distributed local strengths) does not yield correct strength predictions for structures smaller than about one autocorrelation length [54]. The reason for this is that the local material strength must be autocorrelated in reality over a nonzero length and therefore the assumption of the independency of local strengths is incorrect. In the same paper an analytical formula for strength is proposed, supported by theoretical considerations for the strengths of very small and very large structures and numerical simulations using the methodology described in this paper.

An open question is the influence of possible correlations between local material parameters, such as the local strength f_t and fracture energy G_F in the case of concrete. Some results in this regard has already been obtained by Novák et al. [35].

Let us note that both the KLE and EOLE methods offer a random field to be represented in terms of a continuous function (even though it is based on a discrete grid of values in the case of EOLE). This feature is very important for the SFEM in cases when the points where the fields are expanded are not known in advance. This can happen for instance in combination with the adaptivity of FE meshes [55].

8. Conclusions

The main result of this paper is the utilization of the spectral properties (eigen-properties) of defined block correlation matrices. These can be advantageously utilized for the simulation of multivariate stochastic fields with a simple cross correlation structure and a common distribution of components. If all fields share the same distribution shape, the decomposition of the autocovariance structure is done only once for all univariate fields. For Gaussian vector random fields, the resulting distribution and correlation properties are correct. For non-Gaussian fields the autocorrelation structure is correct for all fields, but taking full advantage of the computational simplification brings about small errors in cross correlations. These errors can be predicted without any simulations. The reduction of computational effort is often significant.

The simulation of random functions in the introduced series expansion methods requires generation of random variables; it has been shown how to easily simulate these variables for the representation of cross correlated random fields within the framework of Monte Carlo simulation.

An example in the framework of the SFEM where cross correlated stochastic fields play an interesting role has been outlined. An error assessment procedure for the mean, variance and correlation structures has been presented. It has been shown that a correlation error in simulated random variables significantly

influences both the scatter of the autocorrelation function of expanded random fields and the cross correlation structure.

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