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Incorporation of statistical length scale into Weibull strength theory for composites

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ABSTRACT

In this paper an extension of Weibull theory by the introduction of a statistical length scale is presented. The classical Weibull strength theory is self-similar; a feature that can be illustrated by the fact that the strength dependence on structural size is a power law (a straight line on a double logarithmic graph). Therefore, the theory predicts unlimited strength for extremely small structures. In the paper, it is shown that such a behavior is a direct implication of the assumption that structural elements have independent random strengths. By the introduction of statistical dependence in the form of spatial autocorrelation, the size dependent strength becomes bounded at the small size extreme. The local random strength is phenomenologically modeled as a random field with a certain autocorrelation function. In such a model, the autocorrelation length plays the role of a statistical length scale. The focus is on small failure probabilities and the related probabilistic distributions of the strength of composites. The theoretical part is followed by applications in fiber bundle models, chains of fiber bundle models and the stochastic finite element method in the context of quasibrittle failure.

1. Introduction

In many applications of composites in engineering, an important issue is the statistical strength of the composite. The local tensile strength of fibrous composites can often be expressed as the strength of a bundle of fibers bridging a crack. In such situations, the strength can be studied using fiber bundle models with either local or global load sharing. Typically observed strengths (of fibers) are orders of magnitude below theoretical molecular bonding strengths, which suggests that the observed strengths are primarily the result of flaw types, internal flaws such as voids, inclusions and weak grain boundaries, and external or surface flaws such as notches or cracks, which cause stress concentrations. The reliability of materials depends on these flaws. In this paper we follow the concept of studying the behavior of fibrous materials with the ambition to extend it to other, possibly non-fibrous materials using the argument that various combinations of series and parallel couplings of microbonds with similar behavior to fibrous materials can be found elsewhere. The reliability problems of fibrous materials, such as those studied by Daniels [1] and Coleman [2], have enjoyed many revivals over the last four decades and stimulated considerable interest in the development of new models and asymptotic results [3–8].

Engineering structures made of all kinds of materials must be designed for an extremely low failure probability, such as $P_f = 10^{-6}$–10^{-7} per lifetime (e.g. [27,31,32]). This is necessary to make structural failures very rare compared to other hazards which people face and that are generally accepted. In the range of such extremely low probabilities, the difference between the exponentially decaying Gaussian (normal) distribution and the Weibull distribution, which has a tail decaying as a power law, is enormous, despite both of them having a similar shape in the central region of the distribution that can be estimated from physical or computer experiments. A direct determination of the *tail* of the probability density function (pdf) of a failure load F from experimental histograms is virtually impossible for such low probabilities of failure. The same holds also for computational approaches to complex structures based on direct Monte Carlo simulation.

Even though the current stochastic finite element method (SFEM) has become highly developed [28–30], its extension to extreme value statistics remains a challenge. To compute the tolerable loads of such extremely low P_f, effective SFEMs for extreme value statistics have been developed; they include 'importance sampling' [28,33,34], 'subset simulation' [35,36], 'line sampling' [37], von Neumann's 'splitting', 'Russian roulette' [37], etc. However, despite the development of these powerful methods, their practical application faces a serious obstacle: the results depend strongly on the far-out tail of the probability density function (pdf) of the *input*, though the pdf is typically verified (and can be verified) only for the core, and when this pdf is simply extended into the tail, the extension is often incorrect [38]. If this problem is ignored, any advanced sampling strategy or any other type of SFEM for extreme value statistics is reduced to a mere mathematical exercise of no practical relevance [13].

Therefore, one must rely on a theory Which can be verified indirectly. Its formulation is a fundamental problem of failure

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mechanics, in which only two limiting failure types are now adequately understood [39]. The first is perfectly ductile (plastic) failure, where the failure load is essentially a weighted *sum of the strength contributions* from all the representative volume elements (RVEs) of the material lying on the failure surface, and (because of the central limit theorem of probability) the pdf of the failure load is necessarily Gaussian (except in far-left tails). The second limiting failure type is perfectly brittle failure, which is decided by the *failure of one* RVE and thus follows the weakest-link model, leading to Weibull [13] probability distribution.

This paper, which is an extension of a conference paper [14], starts with the classical Weibull theory (Section 2) employed to explain the length effect on the strength of a single fiber. This concept is extended in Section 2.1 by the incorporation of a length scale. In Section 3, the impact on the strength of the classical fiber bundle is illustrated. Section 4 proceeds to the chain-of-bundles that is supposed to model the strength of a composite structure.

2. Classical Weibull strength theory

Consider a body (structure) under uniform stress containing randomly distributed flaws, see Fig. 1 left. The size of the body is characterized by its length l (e.g. the length of a fiber). The structure fails once the stress at the weakest point (cross section) reaches its local strength. Assume that the local strength is random and characterized by the Weibull distribution (two parametric). Using the weakest-link model together with the Weibull-type function for the concentration of defects, the probability of failure P_f at a given level of stress σ is expressed as the so-called Weibull integral [9]:

$$P_f(\sigma) = 1 - \exp\left[-\int_l \left\langle \frac{\sigma}{s_0}\right\rangle^m \frac{dl}{l_0}\right] \tag{1}$$

where the Malacuya brackets stand for positive part $\langle\bullet\rangle = \max(\bullet, 0)$. The argument in the Malacuya brackets with its power m represents a particular choice of *concentration function.* It represents a contribution to the failure probability of the whole structure. For a given Weibull modulus (shape parameter) m, we have a reference length l_0 with a corresponding scale parameter s_0 of the Weibull strength. The uniform stress level is independent of the position over the length and therefore we can rewrite Eq. (1) as $-\ln(1-P_f) = (\sigma/s_0)^m l/l_0$. Now, the stress level for a chosen probability of failure P_f can be expressed as a function of the structural size (length l):

$$\sigma(l) = s_0(l_0/l)^{1/m}[-\ln(1-P_f)]^{1/m} = s_0 f_W(l)[-\ln(1-P_f)]^{1/m} \tag{2}$$

This function is a power law and therefore its graph in a double logarithmic plot for an arbitrary level of probability P_f (quantile) is a straight line with a decreasing slope of $-1/m$. For example, the mean strength of the structure depends on its length as $\bar{\sigma}(l) = s_0\Gamma(1+1/m)(l_0/l)^{1/m} = s(l)\Gamma(1+1/m)$, where Γ is the Gamma function. The effect of length in this equation and also in Eq. (2) has been inserted into the scale parameter, which then reads $s(l) = s_0(l_0/l)^{1/m} = s_0 f_W(l)$. From here on, we call $f_W(l)$ the Weibull length dependent function. The strength distribution of such a structure is Weibull for an arbitrary length:

$$F(x) = 1 - \exp\left\{-\left[\frac{x}{s(l)}\right]^m\right\} \tag{3}$$

and its shape (parameter m) is independent of the structure size, and the corresponding coefficient of variation (COV) of fiber strength distribution is a constant depending solely on the Weibull modulus m:

$$\mathrm{COV} = \sqrt{\Gamma(1+2/m)/\Gamma^2(1+1/m) - 1} \tag{4}$$

Note that from a Taylor expansion of $F(x)$, we find the lower tail behavior to be $F(x) = [x/s(l)]^m + \mathcal{O}[x^m s(l)]$, where $\mathcal{O}[z]/z \to 0$ as $z \to 0$. This fact will become important in the tail analysis of the strength of fiber bundles. Moreover, numerical evaluation of $F(x)$ necessitates the above approximation of $\exp(x)$ by the Taylor expansion for very small arguments x.

There is a strong relation between the *theory of extreme values* and the *weakest-link model*. Of particular and necessary interest here are the minima of strength realizations over the fiber (chain) length, see Fig. 1 left. It is well known from the theory of extreme values of independent and identically distributed random variables (IID) that there are three and only three possible asymptotic (non-degenerate) limit distributions for minima [10] satisfying the condition $F_n(x) = 1 - [1 - F(x)]^n$. In order to avoid degeneration we look for the linear transformations with the constants a_n and b_n (depending on n) for which the limit distributions $L(x) = \lim_{n\to\infty} L(x) = \lim_{n\to\infty} L_n(a_n x + b_n) = \lim_{n\to\infty}\{1 - [1 - L_n(a_n x + b_n)]^n\}$. Since the Weibull elemental distribution is being considered here, the extremes belong to the domain of attraction of Weibull distribution, and the sequences of constants a_n and b_n satisfying the above recursive relation are known (see e.g. [11]).

The important property readily seen from the above equations is that the scale parameter of the Weibull distribution can be adjusted for any length l_1 to deliver the same P_f as for the original reference length l_0: $s_1/s_0 = f_W(l_1) = (l_0/l_1)^{1/m}$. This demonstrates the inherent feature of the Weibull distribution in the context of the weakest-link model already revealed in Eq. (2): it is arbitrarily scalable with respect to the reference length l_0; there is no length scale inside. Realizing that the reference length of one segment l_1 is arbitrarily scalable, we may perform this randomization with an arbitrary segment length, including the very small $l_1 \to 0$, with the scaling parameter $s_1 \to \infty$, and still obtain the same size effect. The extreme value theory gives us an analytical solution, which was recently proposed for the simplification of computations of large structures with the stochastic finite element method [12,13].

However, the writer believes that the independence assumption of neighboring strengths is not correct for a real spatial distribution of strength in a material and must be abandoned *at a certain length scale*. Also, the *strength must remain bounded* for short segments. The origin of the strength bound is not discussed here, but surely,

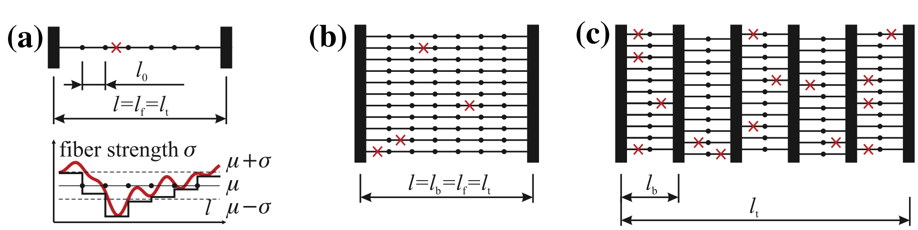

Fig. 1. Unidirectional fibrous tensioned structures with breaks at peak load: (a) one fiber (microbond) discretized into segments of length l_0 with a sketch of a strength random field and its minima; (b) Daniels's bundle of (discretized) fibers; (c) Chain-of-bundles with an illustration of the fragmentation process.

it is not possible to measure arbitrarily high strength with very short specimens. This discrepancy calls for solution.

2.1. Statistical length scale in Weibull strength theory

In order to impose an upper bound on the strength of small structures in the Weibull theory, the independence assumption of any pair of local substructures must be abandoned [15,7]. A plausible and physically acceptable assumption is that neighboring segments of a structure are statistically dependent, while two remote segments are independent. This can be easily modeled by an autocorrelated random field. In other words, it is assumed here that the local strengths are dependent via an autocorrelation function. The autocorrelation can merely be a function of the Euclidean norm of two points; moreover, it can be isotropic, i.e. the autocorrelation can be independent of direction. An example of such a function can be the squared exponential function (power $p = 2$): $R(\Delta d) = \exp[-(|\Delta d|/l_\rho)^p]$. In the model, the strength random field is homogeneous and isotropic, meaning simply that the local distribution is identical in all points of the structure. To remain consistent with the previous text, the strength is assumed to be Weibull distributed from here on. In addition, the relation between the pair of reference shape and scale parameter of the distribution and the autocorrelation length must be formulated explicitly. The reason is that the simple scaling relation $s_1/s_0 = f_W(l_1) = (l_0/l_1)^{1/m}$ does not hold anymore. Why? Because a *statistical length scale* in the form of the autocorrelation length l_ρ has been incorporated. As a consequence, the strength dependence upon the size (length) is not a power law anymore. The autocorrelation has the effect of imposing an upper bound on strength for infinitely small (short) structures. When the structural size converges to zero, the weakest-link mechanism disappears and the strength is identical to the elemental distribution (the highest attainable strength of the model at the currently modeled scale such as micro, meso, macro). By adding more material (increasing length), the weakest-link effect gradually overtakes and causes the decrease of structural strength (both the mean strength and also its standard deviation are reduced). At the limit, one can show that the large size asymptotic behavior is the classical Weibull size effect. In other words, for very large structures the effect of relatively small autocorrelation length becomes insignificant and the model can again be treated as the weakest-link model of independent identically distributed random strength elements. The crossover length is the autocorrelation length. To conclude, the fiber strength has the same form as in Eq. (2), but with a different length dependent function. In particular, a smooth interpolation function proposed recently in [16,15,7] has correct asymptotes: the left asymptote at the small size limit is horizontal and the right asymptote is the classical Weibull function $f_W(l)$ from Eq. (2):

$$f_V(l) = \left(\frac{l_\rho}{l + l_\rho}\right)^{1/m} \tag{5}$$

At large sizes, self-similar behavior is recovered (the double logarithmic plot is a straight decreasing line with a slope of $-1/m$). At small sizes, the weakest-link mechanism is suppressed by the fact that all substructures share an identical strength due to their perfect positive dependence. Note that this relation is supported by numerical simulations of extremes (minima) of random fields and is an alternative to currently available analytical results [17]. Note also that the shape (m or COV) of the distribution remains independent of size l in Eq. (2). An illustration of the mean fiber size effect exploiting Eq. (5) is provided in Fig. 2 left, with comparison to the classical Weibull dependence.

3. Strength of fiber bundles

The above - described extension of Weibull theory can be readily incorporated into the theory of the strength of bundles with elastic-brittle fibers and with global load sharing [15,7]. The classical model formulated by Daniels [1] describes a situation where n parallel fibers (or microbonds) with IID random strengths, equal lengths and elastic moduli, are stretched between two clamps under global load sharing. The maximum tensile force of the bundle Q_n^* is measured in terms of load per fiber. Daniels [1] derived a recursive formula for computing the cumulative density function (CDF) $G_n(x)$ of Q_n^* depending on the fiber CDF $F(x)$ and number of fibers n:

$$G_n(x) = P(Q_n^* \leqslant x) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} [F(x)]^k G_{n-k}\left(\frac{nx}{n-k}\right),$$
$$\text{where} \quad G_1(x) \equiv F(x), \quad G_0(x) \equiv 1 \quad \text{and} \quad \binom{n}{k} = \frac{n!}{(n-k)!k!} \tag{6}$$

This formula is usable only for a small number of parallel fibers (a few tens) as the computational demands and also the round-off errors increase significantly with an increasing number of fibers. Moreover, Daniels proved that, under broad assumptions on $F(x)$, the asymptotic distribution of the maximum bundle load Q_n^* is Gaussian, i.e. with $n \to \infty$, it tends to

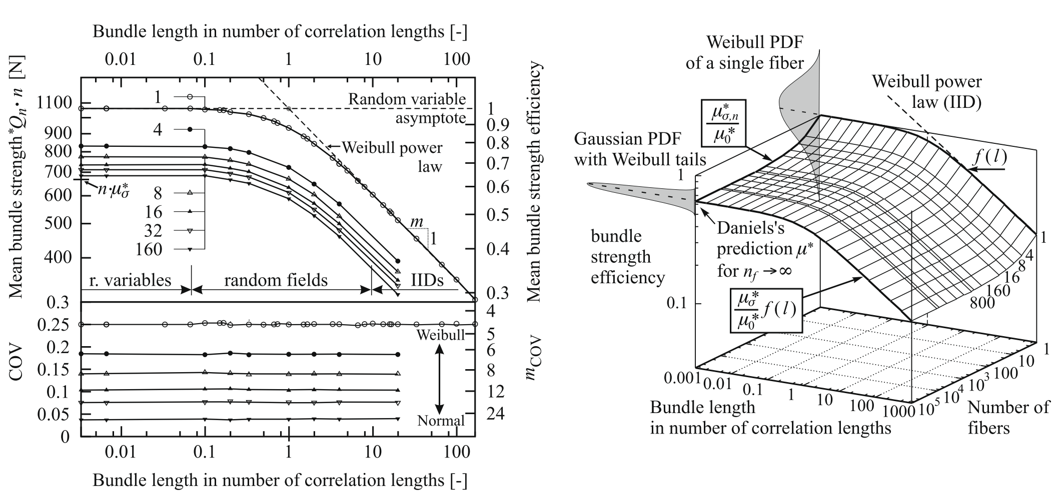

Fig. 2. Left top: mean size effect curves for an increasing number of fibers n in a bundle for fiber strength described by Weibull random process, $m = 4.52$. The curves nearly overlap for $n > 160$. Left bottom: effective Weibull modulus m, Eq. (4). Right: 3D representation of bundle efficiency *depending* on the number of fibers and bundle length for the case of random strength described by random processes.

$$\lim_{n\to\infty} G_n(x) = \Phi\left(\frac{x-\mu^*}{\gamma^*}\sqrt{n}\right) = \Phi\left(\frac{x-\mu^*}{\sigma^*}\right) \tag{7}$$

where $\Phi(\bullet)$ is the standard Gaussian cumulative density function, see Fig. 3 top. The mean value μ^* depends on fiber $F(x)$ and not on the number of fibers n. The standard deviation σ^* of bundle strength is proportional to the inverse of the square root of n, (see e.g. [7] for details). Formulas for the mean value μ^* and standard deviation σ^* in the case of Weibull strength distribution $F(x)$ of fibers will be given in Eq. (16).

It can be argued [15,7] that if the CDF of fibers is Weibullian, the left tail of $G_n(x)$ must be nothing but Weibullian, i.e. the Daniels's result about the normality of $G_n(x)$ is valid only in a certain core part of the distribution. Indeed, Harlow et al. [18] showed in 1983 using the set theory that the left tail of $G_n(x)$ is Weibullian with the shape parameter (Weibull modulus) being an n-multiple of the shape factor m of one fiber. Recently, Bažant and Pang [8] confirmed this result and they have shown that the result holds also for ductile fibers.

The bundle strength distribution $G_n(x)$ over the whole range of strengths x is, in fact, a smooth transition between two tail Weibull distributions $W_0(x)$ and $W_\infty(x)$. Practically however, most of the distribution transition is occupied by the above Gaussian distribution derived by Daniels, see Eq. (7). The spread of the Gaussian core is usually from loads corresponding to probabilities ranging from enormously small to enormously large. The Weibull distribution for small strengths (or high reliabilities) $W_0(x)$ reads:

$$\lim_{x\to 0} G_n(x) = W_0(x) = 1 - \exp\left[-\left(\frac{x}{S_n^*}\right)^{nm}\right] \approx \left(\frac{x}{S_n^*}\right)^{nm} \tag{8}$$

where the scale parameter comes out of the recursion in Eq. (6) in which $F(x)$ is set to $(x/s)^m$:

$$S_n^* = \frac{s(l)}{(d_n^*)^{\frac{1}{nm}}}, \quad d_n^* = (-1)^{n+1} + \sum_{k=1}^{n-1}(-1)^{k+1}\binom{n}{k} d_{n-k}^*\left(\frac{n}{n-k}\right)^{m(n-k)} \tag{9}$$

The fiber shape parameter $s(l)$ is assumed to be known for a given length l, see Section 2 and the remainder of this subsection. The Weibull distribution for large strengths $W_\infty(x)$ reads:

$$\lim_{x\to\infty} G_n(x) = W_\infty(x) = 1 - \exp\left[-n\left(\frac{x}{s(l)}\right)^m\right] \tag{10}$$

This is derived from the weakest-link model and the associated extreme value distribution. At high loads, the bundle strength per fiber is no better than its weakest fiber:

$$1 - W_\infty(x) = [1 - F(x)]^n = \left\{\exp\left[-\left(\frac{x}{s(l)}\right)^m\right]\right\}^n \tag{11}$$

From Eq. (11) the above Eq. (10) follows easily. The upper tail $W_\infty(x)$ is, however, of low importance for structural reliability considerations. An example of $G_n(x)$ for $n = 20$ fibers is provided in Fig. 4 together with a comparison of the three distributions and strength distribution of bundle with a single fiber ($m = 4$). The solid circle in Fig. 4 right is the intersection of the Weibull asymptotical distributions.

For practical numerical computations and implementation into software packages, a distribution composed of two branches might suffice. Such a distribution has recently been suggested in [8]. The intersection between the Gaussian core and the left Weibull tail (see the solid box in Fig. 4) can be used as a point for the grafting of the Weibull tail $W_\infty(x)$ onto a scaled Gaussian core in which the scaling factor is calculated to achieve unit probability for infinite strength x.

The effect of parallel coupling seems to be captured well. The question remains as to what the *effect of the length* of such a bundle on its strength is. It has been shown in [15,7] that, for the situation studied, the effect of length and parallel coupling can be treated separately, and they are independent and do not interact. Simply, a change in the length of the fibers in the bundle results only in the change in the scaling parameter s of the distribution $F(x)$ of fiber strength. This distribution then enters formulas for the bundle strength distribution. If, for example, we consider Weibull fibers, and, in analogy with Eq. (2), we use the association of the length dependence with the scale parameter $s(l) = s_0 f(l)$, the resulting Weibull strength distribution can be plugged into Daniels's formulas for bundle strength. After a few simple manipulations [15,7], the resulting mean bundle strength reads $\mu^*(n, l) = \mu^*(n) f(l)$, thus manifesting the decomposed effects of length and parallel coupling. The bundle strength, being a function of the amount of

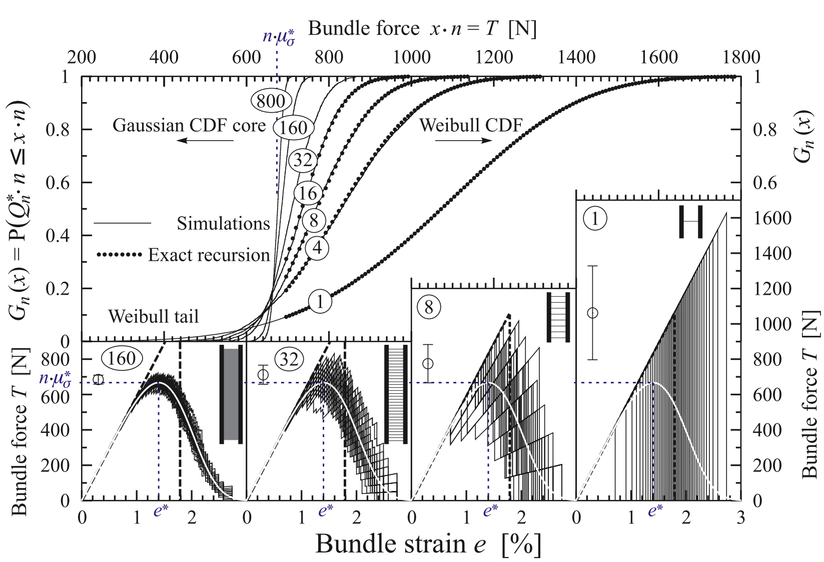

Fig. 3. Top: bundle strength distribution with a growing number of fibers n obtained either using Daniels's exact recursion for small n or using direct Monte Carlo simulation. Bottom: samples of the whole force–strain diagrams obtained by Monte Carlo simulations of the bundle response for selected n in the bundle (note the quasibrittleness induced solely by the statistical scatter of strength). Bundles are sketched and the mean value of strength ± standard deviation is marked by a circle with errorbars. The white lines show the asymptotic mean force–strain diagram (see [15,7] for details).

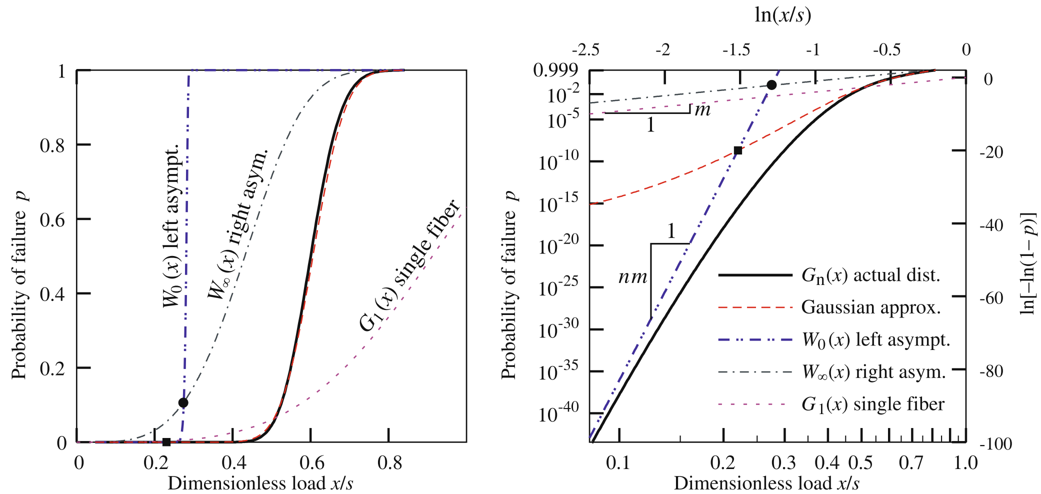

Fig. 4. Cumulative distribution function of bundle strength. Comparison of a bundle with $n = 20$ fibers and a single fiber bundle. Left: linear plot. Right: Weibull probability paper.

material (fiber length and number of parallel fibers), is plotted in Fig. 2. The figure compares the proposed incorporation of the statistical length scale l_ρ using $f_V(l)$ with the classical Weibull theory that uses $f_W(l)$. It is shown that with an increasing number of fibers (or microbonds), the crossover length l_ρ propagates in the size effect plots unchanged.

4. Strength of chains of fiber bundles

The above theory presented in the previous section applies only to fibrous structures with non-interacting fibers (a fiber continuum at the limit) or generally to *tension in frictionless materials*. In fiber reinforced composites or other materials, a certain shear transfer of load takes place to neighboring real or virtual fibers. Such a transfer can have various origins and forms. For example, when fibrous materials are loaded in tension, the effect of matrix (in early models of composites), yarns with slight twist or cables with periodic friction clamps is to localize the failure of one fiber (microbond) in such a way that the load is lost only near the break. The lost load is carried by the surviving fibers or bond in the vicinity of the break. This situation can be modeled by the so-called local load sharing rules. In general, an accepted model for the failure mechanism is the *chain-of-bundles* model [3]; the material is assumed to behave like a chain of short bundles of common length l_b. From here on, we ignore the local load sharing rule and continue to model the stress redistribution in each bundle by the global load sharing rule. The structure under study is sketched in Fig. 1c. Even though such a model can not reproduce all of the effects of shear load transfer, for the study of the most influential mechanisms of size effect this statistical model suffices.

The common length l_b (sometimes called the effective length) is assumed to remain constant during loading (even though this assumption may not be realistic especially for twisted yarns during transition from low loads with small transverse pressure to high loads, because the effective length is sensitive to lateral pressure). The basic questions answered here are: (i) what the limiting form of the strength probability distribution is and (ii) what the effect of modeling the local fiber random strength by an autocorrelated random field is. These results may be extremely important in the study of the reliability of materials and may serve as a basis for effective homogenization techniques for the modeling of materials which, for reliability purposes, must be based on higher order statistical moments of the random mechanical response.

The strength of the chain-of-bundles model is governed by the weakest bundle. By assuming bundles to be independent (or at least the distance of bundles must be more than about five times greater than the autocorrelation length of the fiber random strength field), one can formulate the strength distribution as:

$$H_{k,n} = 1 - [1 - G_n(x)]^k \tag{12}$$

For Weibull fibers the left tail is Weibullian with the same shape factor (modulus) as the left tail of $G_n(x)$, which is mn. Note that the importance of the left tail ($x \downarrow 0$) of the strength distribution, which can be approximated as $G_n(x) \approx W_0(x) \approx [x/s_n^*(l)]^{mn}$, $s_n^*(l) = f(l)s_n^*$, grows with k due to the weakest-link mechanisms. The left tail of a single bundle strength is Weibullian and therefore the distribution $H_{k,n}$ consists of a Weibullian left tail. After a simple algebra, one can approximate the distribution to be (for large k and small n):

$$H_{k,n}(x) \cong 1 - \exp\left\{-k\left[\frac{x}{s_n^*(l)}\right]^{nm}\right\} \tag{13}$$

which is a Weibull distribution with the shape parameter mn (equal to the shape parameter of a single bundle) and scale parameter $k^{-1/(km)}s_n^*(l)$. For large k and large n the distribution of strength of a single bundle is almost entirely Gaussian. The Weibull left tail $W_0(x)$ is extremely short, see e.g. the solid box in Fig. 4 right at the probability of 2×10^{-9}. Therefore, most of the distribution of strength of the chain-of-bundles tends to Gumbel distribution with increasing k (Gumbel distribution is the limiting form of extremes of Gaussian iids). Indeed, Smith (see Theorem 6.1 in [19], also [20,21]) proposed the approximation of the CDF in the form:

$$H_{k,n}(x) \cong 1 - \exp\left[-\exp\left(\frac{x - b_{k,n}}{a_{k,n}}\right)\right] \tag{14}$$

where the constants $a_{k,n}$ and $b_{k,n}$ can be easily calculated from the two moments (mean and standard deviation μ, σ) of the Gaussian strength of one bundle using standard formulas from the extreme value theory [22]:

$$a_{k,n} = \frac{\sigma^*}{\sqrt{2w}}, \quad b_{k,n} = \mu^* + \sigma^*\left[\frac{\ln(w) + \ln(4\pi)}{\sqrt{8w}} - \sqrt{2w}\right] \tag{15}$$

where $w = \ln(k)$. The *mean value* of this Gumbel distribution reads simply $\mu_k \approx b_{k,n} - 0.577a_{k,n}$ ($0.577 \approx$ Euler–Mascheroni constant), the median equals $b_{k,n} + a_{k,n}\ln[\ln(2)]$ and the standard deviation

$\sigma_k = \pi a_{k,n}/\sqrt{6}$. The experience with small k (say 2–300) is that the approximation slightly overestimates the mean value, for which the writer proposes a polynomial approximation: $\mu_k = \mu^* + \sigma^*(-0.007w^3 + 0.1025w^2 - 0.8684w)$. It is concluded here that, for a sufficiently large number k of bundles in a series, $H_{k,n}(x)$ is best formulated as a transition from Weibull tail to Gumbel core, similarly to the situation for a single bundle where $G_n(x)$ is a transition from Weibull tail to Gaussian core.

Let us proceed with the effect of the assumption of spatially correlated strengths of elements of fibers in the chain-of-bundles structure. It is known from experiments on twisted yarns that a small twist can increase the yarn's strength. It is believed that the reason is the so-called *fragmentation effect*: a fiber in twisted yarn acts over several lengths l_b and its breaking affects only the bundle surrounding the break (ineffective length); the fiber is capable of supporting almost the original load at only a short distance from the break. Therefore one fiber can break several times over its length, which may lead to an overall increase in the strength of the structure [6], see Fig. 1c for an illustration.

The remainder of this section studies the strength of such a chain-of-bundles model with elastic-brittle fibers of varying total lengths l_t and with a varying number of bundles k inside it. Throughout the study, the material parameters are kept constant; in particular, the Weibull fibers have the shape parameter m, and the scale parameter $s = s_\rho$ is associated with the reference length l_ρ (the reference length has the meaning of the autocorrelation length in the case of a random strength field). The number of fibers must be specified as it influences the variability of bundle strength and thus also the slope of the size effect curve (bundles are arranged in series). For the numerical study we select a medium size of $n = 160$.

As mentioned before, the correlation length l_ρ is kept constant. Let us express the total specimen length as an n_ρ-multiple of the autocorrelation (reference) length: $l_t = n_\rho l_\rho$. This total specimen length can be divided into various numbers k of bundles each having the length l_b: $l_t = kl_b$. Now, it suffices to study the effect of n_ρ and k on the strength of the final composite (chain-of-bundles). The Gaussian core reaches extremely low probabilities in the case of $n = 160$. Therefore, the mean strength will be the mean of a minimum of k Gaussian distributed variables representing the strength of one bundle. To compute this mean, one can use the aforementioned formulas from the paragraph following right after Eq. (15). The Gaussian variables have the mean value μ^* and standard deviation σ^* given in [1] and refined in [19], see [7]. The formulas are rearranged here to be explicitly dependent on the length function $f(l_b)$, for which there are two alternatives (the Weibull form f_W and the proposed formula f_V based on extremes of random processes):

$$\mu^* = f(l_b)\mu_\rho = f(l_b)\underbrace{s_\rho\left[m^{-1/m}c_m + 0.996n^{-2/3}m^{(-1/m-1/3)}\exp\left(-\frac{1}{3m}\right)\right]}_{\mu_\rho(m,n,s)\ldots\text{for the reference length}}$$

$$\sigma^* = f(l_b)\sigma_\rho = f(l_b)\underbrace{s_\rho\left[m^{-1/m}\sqrt{c_m(1-c_m)}/\sqrt{n}\right]}_{\sigma_\rho(m,n,s)\ldots\text{for the reference length}},\quad c_m = \exp(-1/m) \tag{16}$$

Both studied fiber length dependent functions can be rewritten in non-dimensional variables used as arguments of our final graphs (factor out the correlation length l_ρ):

$$f_W(l_b; m, l_\rho) = \left(\frac{k}{n_\rho}\right)^{1/m},\quad f_V(l_b; m, l_\rho) = \left(\frac{l_\rho}{l_\rho + l_b}\right)^{1/m} = \left(\frac{k}{k+n_\rho}\right)^{1/m} \tag{17}$$

The mean strength of a composite for small k can be approximated as $\mu_k = f(l_b)[\mu_\rho + \sigma_\rho(-0.007w^3 + 0.1025w^2 - 0.8684w)]$, $w = \ln(k)$. For large k, one can use Eq. (15). In both cases, the effect of length on the mean strength of a composite can be written as a multiple of a solution for the reference length. The mean strength of one bundle with 160 fibers is again ca 64% of the mean strength of a reference fiber ($m = 4.52$); compare with Fig. 2.

Deviations of bundle and composite strength from the fiber strength disappear with growing Weibull modulus m (as it approaches the deterministic situation). Pan [6] has used the fragmentation effects to explain the strength growth in twisted yarns with a light twist. He used the Weibull strength dependency (see Fig. 5) and argued that the mean strength of twisted yarn can exceed the fiber reference mean strength. The writer believes that this is an incorrect conclusion. Series coupling of bundles and the associated weakest-link mechanism must have a stronger effect on strength than shortening the fiber length l_b in each bundle (shortening the length leads to a statistical strength increase). These two effects cancel each other out for $n = 1$ fiber. Unfortunately, Pan's analysis also disregards the number of fibers in one bundle, which has an impact on bundle strength variance. Note that a large strength variability at the bundle level has a strong impact on the size effect strength (chain of bundle strength), because of the weakest-link mechanism. It can be seen from Fig. 5, left, that the proposed solution for the composite mean strength becomes slightly lower than the bundle strength with growing k. This decrease becomes pronounced with lower numbers of fibers n and also with greater material strength variability (smaller m).

The presented results can be used to explain the dependence of the strength of twisted yarns on the twist level. If the length of the composite is 1–10 times greater than a correlation length, yarns exhibit a slight increase in the mean strength for a medium twist

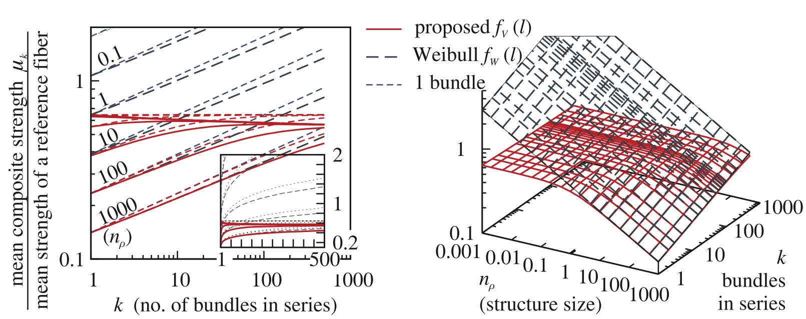

Fig. 5. Mean composite or *bundle* strength (chain-of-bundles) for $n = 160$ fibers plotted with logarithmic coordinates. Comparison of Weibull length dependency with the proposed Eq. (5). Left: cuts of surfaces for selected n_ρ and a comparison with the mean strength of one bundle (the inset has linear coordinates with the same ranges of axes). Right: 3D representation.

level. For greater twists, the experimentally reported strength drop can be explained by Phoenix's [23] theory: fibers are assumed to follow helical paths with the helix angle α, and the bundle strength is then proportional to $\cos^2 \alpha$. In other words, Phoenix's model predicts the strength of yarn to achieve its maximum for zero twist and decrease with an increasing amount of twist. Note that Phoenix [23] assumed non-interacting fibers. In our model, the interaction of fibers in real lateral pressure-sensitive yarns is introduced through using the chain-of-bundles model instead of using just a fiber bundle model. The combination of (i) Phoenix's [23] theory with (ii) the presented stabilization of the Weibull theory by the statistical length scale in the chains of fiber bundle model captures the experimentally observed [40] slight strength increase of yarns with a light twist that is followed by a drastic strength decrease for highly twisted yarns.

5. Discussion and relations to the strength of composites and quasibrittle structures in general

In a large fiber composite, several clusters of fiber damage occur during loading and the "weakest" cluster is responsible for the final demise of the composite. The failure process is localized in nature and it is believed that as the number of fibers in the composite (or parallel microbonds in a general material) is increased, one can identify a region within a composite that is statistically representative of the rest. A large system – a composite – can be formally considered as composed of a collection of independent subsystems coupled in series so that a failure in the weakest subsystem causes a failure across the entire system. The composite is then as strong as the weakest of these regions and this leads to the conception of a *critical cluster*. As a result, large composites obey the weakest-link rules and the structural strength is associated with extreme value statistics.

The statistical description of the strength and mechanical properties of a cluster depend on the size of the critical region. Under the assumptions presented in the previous sections, the strength distribution has a Weibull left tail (important for the reliability of the composite material in practical applications) and a Gaussian to Gumbel core. The size of a critical cluster l_s may then be, under similar loading conditions, seen as a material property stemming from the variability of material properties and the mechanics of the failure of the critical cluster. The question is what the dependencies and interactions between such clusters are. In the fracture mechanics of concrete, it is now widely believed that the *characteristic length* (a material constant) related to the size of the fracture process zone (FPZ) has a connection with the maximum aggregate size. For geometrically similar structures of different sizes, the FPZ is approximately constant, and this implies the so-called energetic-deterministic size effect [24]. Another explanation of the nominal strength dependence on the size is the classical statistical size effect caused by strength variability and weakest-link mechanisms. These two sources can be treated as independent and modeled together in a complex model [12,25]. However, based on the above discussion, there may be a relation between the FPZ size, the characteristic size based on the maximum aggregate size the size of the critical cluster of cracks and finally the autocorrelation size of the local random strength field introduced at the beginning of the present paper.

The described concept of a random cluster strength field has been utilized in stochastic finite element computations of concrete structures in higher dimensions, where the effective properties at the macro-scale were assumed to be spatially correlated over a certain constant length l_s [26]. It can be shown that the deterministic characteristic scaling length (incorporated into the model e.g. via the crack band size) interacts with the statistical scaling length (introduced in the form of autocorrelation length as explained above). Namely, two examples of application can be mentioned [12,25]; the former aims at numerically studying the interplay of several sources of size effects of concrete dog-bone specimens of different sizes (ratio 1:32) loaded in tension, and the results are compared to the result of real experiments. The latter application constitutes support for the formulation of a size effect formula for strength for structures failing after crack initiation from a smooth surface. The formula exploits the above - explained upper bound on strength attained for small structural sizes. The paper involves the Malpasset Dam as an example of a real structure. The theoretical arguments supported by extensive numerical simulations have lead to the formulation of a procedure based on the new size effect formula. The formula has been proposed for the prediction of the complex energetic-statistical size effect (based just on deterministic nonlinear computation and a single analysis of the Weibull integral) without performing advanced and time consuming stochastic nonlinear simulations [12].

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