

APLIKACE DVOJNÉHO INTEGRÁLU

Př: Vypočítejte obsah rovinného obrazce:

1, $A: x^2 + y = 4, x + y = 2$

2, $A: y \leq 1, y \leq x-1, y \geq \ln x$

3, $A: xy = 4, x + y = 5$

4, $A: x^2 + y^2 = 4x, x^2 + y^2 = 2x, y = x, y = 0$

Př: Vypočítejte objem tělesa:

1, $W: x + y + z = 6, 3x + 2y = 12, y = 0, z = 0$

2, $W: z = x^2 + y^2, x + y = 1, x = 0, y = 0, z = 0$

Př: Vypočítejte obsah části plochy:

1, $S: 6x + 3y + 2z = 12, x \geq 0, y \geq 0, z \geq 0$

2, $S: 2z = x^2, y \leq 2x, y \geq \frac{1}{2}x, x \leq 2\sqrt{2}$

Př: Určete těžiště tenké rovinné desky:

1, $A: x = 1, y = 0, y = \sqrt{x}; \sigma(x, y) = \frac{1}{x+1}$

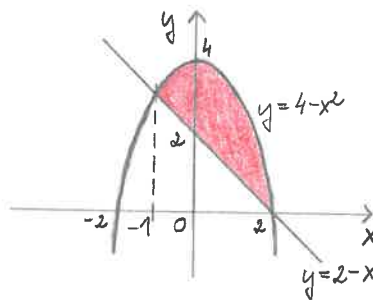
APLIKACE DVOJNÉHO INTEGRALU

Př.: Vypočítejte obsah rovinného obrazce.

1) A: $x^2 + y = 4$, $x + y = 2$ ← přímka

parabola $y = 4 - x^2$ $y = 2 - x$

$$\begin{aligned} 4 - x^2 &= 2 - x \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0 \\ x_1 &= -1, x_2 = 2 \end{aligned}$$



oblast I. druhu:

$$\begin{aligned} -1 &\leq x \leq 2 \\ 2 - x &\leq y \leq 4 - x^2 \end{aligned}$$

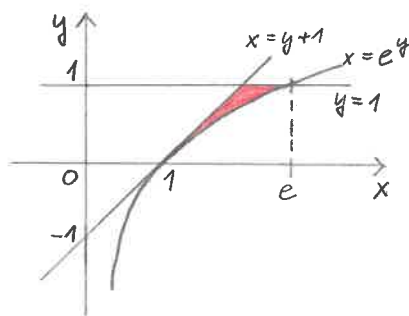
$$P = \iint_A dx dy = \int_{-1}^2 \left[\int_{2-x}^{4-x^2} dy \right] dx =$$

$$= \int_{-1}^2 \left[y \right]_{2-x}^{4-x^2} dx = \int_{-1}^2 (4 - x^2 - 2 + x) dx = \int_{-1}^2 (2 + x - x^2) dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^2 =$$

$$= 4 + 2 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3} = 8 - \frac{9}{3} - \frac{1}{2} = 5 - \frac{1}{2} = \underline{\underline{\frac{9}{2}}}$$

2) A: $y \leq 1$, $y \leq x - 1$, $y \geq \ln x$
 $x = y + 1$ $x = e^y$

Pozn: Přímka $y = x - 1$ je
 tečnou křivky $y = \ln x$
 v bodě $[1, 0]$.



oblast II. druhu:

$$\begin{aligned} 0 &\leq y \leq 1 \\ y + 1 &\leq x \leq e^y \end{aligned}$$

$$P = \iint_A dx dy = \int_0^1 \left[\int_{y+1}^{e^y} dx \right] dy = \int_0^1 [x]_{y+1}^{e^y} dy = \int_0^1 (e^y - y - 1) dy =$$

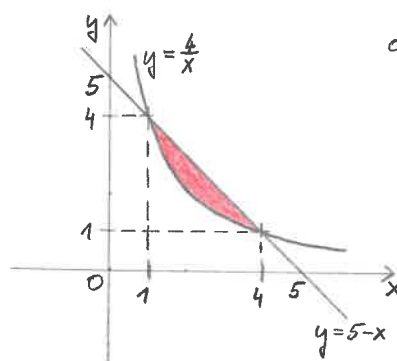
$$= \left[e^y - \frac{y^2}{2} - y \right]_0^1 = e - \frac{1}{2} - 1 - \underbrace{1}_{e^0} = \underline{\underline{e - \frac{5}{2}}}$$

3) A: $x \cdot y = 4$, $x + y = 5$
 $y = \frac{4}{x}$ $y = 5 - x$
 hyperbola $\frac{4}{x} = 5 - x \quad | \cdot x$

$$x^2 - 5x + 4 = 0$$

$$(x-1)(x-4) = 0$$

$$x_1 = 1, x_2 = 4$$



oblast I. i II. druhu:

$$1 \leq x \leq 4$$

$$\frac{4}{x} \leq y \leq 5 - x$$

$$P = \iint_A dx dy = \int_1^4 \left[\int_{\frac{4}{x}}^{5-x} dy \right] dx =$$

$$= \int_1^4 \left[y \right]_{\frac{4}{x}}^{5-x} dx = \int_1^4 \left(5 - x - \frac{4}{x} \right) dx = \left[5x - \frac{x^2}{2} - 4 \ln|x| \right]_1^4 =$$

$$= 20 - 8 - 4 \ln 4 - 5 + \frac{1}{2} + 4 \ln 1 = \frac{15}{2} - 8 \ln 2$$

$\ln 2^2 = 2 \ln 2$

4) A: $x^2 + y^2 = 4x$, $x^2 + y^2 = 2x$, $y \leq x$, $y \geq 0$

$$(x-2)^2 + y^2 = 4$$

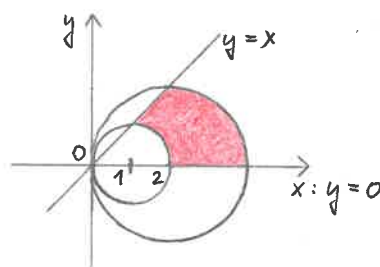
$$\rightarrow r^2 = 4r \cos \varphi$$

$$r = 4 \cos \varphi$$

$$(x-1)^2 + y^2 = 1$$

$$\rightarrow r^2 = 2r \cos \varphi$$

$$r = 2 \cos \varphi$$



Transformace
do polárních
souřadnic:

$$0 \leq \varphi \leq \frac{\pi}{4}$$

$$2 \cos \varphi \leq r \leq 4 \cos \varphi$$

$$P = \iint_A dx dy = \int_0^{\frac{\pi}{4}} \left[\int_{2 \cos \varphi}^{4 \cos \varphi} r dr \right] d\varphi =$$

$$= \int_0^{\frac{\pi}{4}} \left[\frac{r^2}{2} \right]_{2 \cos \varphi}^{4 \cos \varphi} d\varphi = \frac{1}{2} \int_0^{\frac{\pi}{4}} (16 \cos^2 \varphi - 4 \cos^2 \varphi) d\varphi = 6 \int_0^{\frac{\pi}{4}} \cos^2 \varphi d\varphi =$$

$$= 3 \int_0^{\frac{\pi}{4}} (1 + \cos 2\varphi) d\varphi = 3 \left[\varphi + \frac{1}{2} \sin 2\varphi \right]_0^{\frac{\pi}{4}} =$$

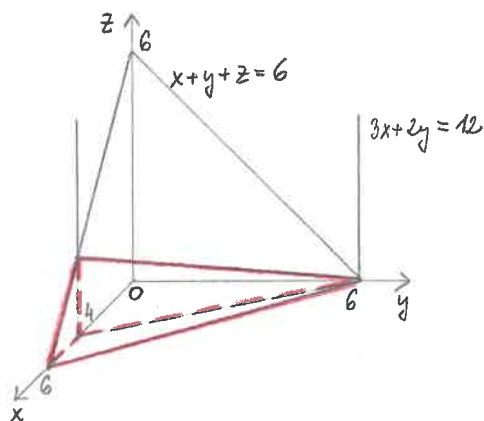
$$= 3 \left(\frac{\pi}{4} + \frac{1}{2} \sin \frac{\pi}{2} - 0 - \frac{1}{2} \sin 0 \right) = 3 \left(\frac{\pi}{4} + \frac{1}{2} \right) = \frac{3}{4} (\pi + 2)$$

Př: Vypočítejte objem válcového tělesa.

1) $W: x+y+z=6, 3x+2y=12, y=0, z=0$

$\rightarrow$ rovina
 $\left\{ \begin{array}{l} \text{průsečnice s } z=0: \\ x+y=6 \\ x=6-y \end{array} \right.$
 $x=4-\frac{2}{3}y$
 $\rightarrow$ rovina kolmá k (x,y)

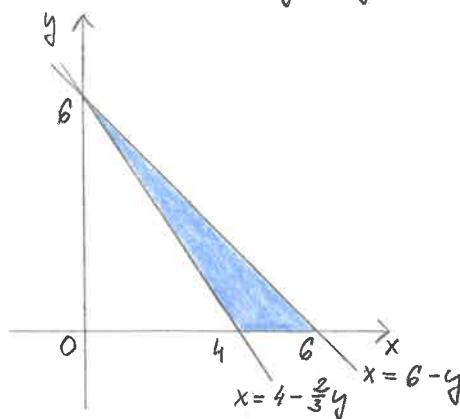
$\bullet f: z=6-x-y$
 $\bullet g: z=0$



Průmět do roviny $(x,y) \rightarrow$ množina A:

oblast II. druhu

$0 \leq y \leq 6$
 $4-\frac{2}{3}y \leq x \leq 6-y$



$$\begin{aligned}
 V &= \iint_A (f(x,y) - g(x,y)) dx dy = \int_0^6 \left[\int_{4-\frac{2}{3}y}^{6-y} (6-x-y) dx \right] dy = \int_0^6 \left[6x - \frac{x^2}{2} - xy \right]_{4-\frac{2}{3}y}^{6-y} dy = \\
 &= \int_0^6 \left(6(6-y) - \frac{1}{2}(6-y)^2 - y(6-y) - 6(4-\frac{2}{3}y) + \frac{1}{2}(4-\frac{2}{3}y)^2 + y(4-\frac{2}{3}y) \right) dy = \\
 &= \int_0^6 \left(36 - 6y - 18 + 6y - \frac{1}{2}y^2 - 6y + y^2 - 24 + 4y + 8 - \frac{8}{3}y + \frac{2}{9}y^2 + 4y - \frac{2}{3}y^2 \right) dy = \\
 &= \int_0^6 \left(2 - \frac{2}{3}y + \frac{1}{9}y^2 \right) dy = \left[2y - \frac{1}{3}y^2 + \frac{1}{54}y^3 \right]_0^6 = 12 - \frac{36}{3} + \frac{6^3}{54} = \\
 &= 12 - 12 + \frac{6 \cdot 6 \cdot 6}{6 \cdot 3 \cdot 3} = \underline{\underline{4}}
 \end{aligned}$$

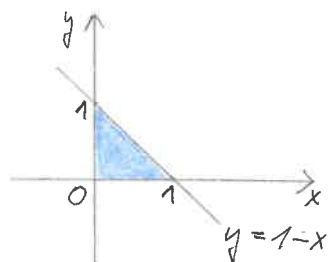
Pozn: 1. Souřadnicové roviny: $(x,y) \dots z=0 \dots$ půdorysna
 $(x,z) \dots y=0 \dots$ nárysna
 $(y,z) \dots x=0 \dots$ bokorysna

2. Těleso W je shora a zdola ohraničeno plochami $f(x,y)$ a $g(x,y)$ - v zadání příkladu je známe tak, že v jejich rovnici vyskytuje proměnná z .

2) $W: \underbrace{z = x^2 + y^2}_{f(x,y)}, \underbrace{x+y=1}_{\text{rovina kolmá k } (x,y)}, \underbrace{x=0}_{g(x,y)}, \underbrace{y=0}_{g(x,y)}, \underbrace{z=0}_{g(x,y)}$

rotací eliptický
paraboloid s osou z

Průřez do roviny $(x,y) \rightarrow$ množina A



oblast I. i II. druhu

$$\begin{aligned} &0 \leq x \leq 1 \\ &0 \leq y \leq 1-x \end{aligned}$$

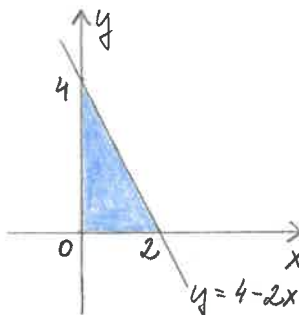
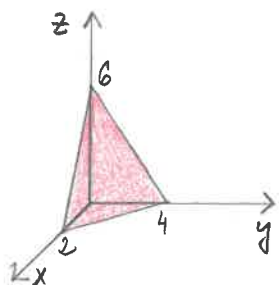
$$\begin{aligned} V &= \iint_A \underbrace{(x^2 + y^2)}_{f(x,y)} \underbrace{- 0}_{g(x,y)} dx dy = \int_0^1 \left[\int_0^{1-x} (x^2 + y^2) dy \right] dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{1-x} dx = \\ &= \int_0^1 \left(x^2(1-x) + \frac{1}{3}(1-x)^3 \right) dx = \int_0^1 \left(x^2 - x^3 + \frac{1}{3} - x + x^2 - \frac{1}{3}x^3 \right) dx = \\ &= \int_0^1 \left(\frac{1}{3} - x + 2x^2 - \frac{4}{3}x^3 \right) dx = \left[\frac{1}{3}x - \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 \right]_0^1 = \\ &= \frac{1}{3} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} = \frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \underline{\underline{\frac{1}{6}}} \end{aligned}$$

Pr: Vypočítejte obsah části plochy.

1) $S: 6x + 3y + 2z = 12, x \geq 0, y \geq 0, z \geq 0$

$$\begin{aligned} &\rightarrow \frac{x}{2} + \frac{y}{4} + \frac{z}{6} = 1 \\ &\rightarrow f: z = 6 - 3x - \frac{3}{2}y \\ &f'_x = -3, f'_y = -\frac{3}{2} \end{aligned} \quad \begin{aligned} &\rightarrow \text{průsečnice s } (x, y): \\ &z = 0 \Rightarrow 6x + 3y = 12 \\ &y = 4 - 2x \end{aligned}$$

Průřez do $(x, y) \rightarrow$ množina A :



Oblast I. i II. druhu

$$\begin{aligned} &0 \leq x \leq 2 \\ &0 \leq y \leq 4 - 2x \end{aligned}$$

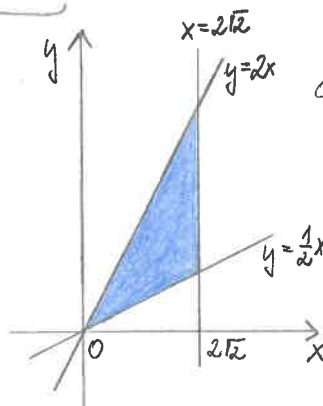
$$\begin{aligned} S &= \iint_A \sqrt{1 + \underbrace{(-3)^2}_{(f'_x)^2} + \underbrace{\left(-\frac{3}{2}\right)^2}_{(f'_y)^2}} dx dy = \iint_A \frac{5}{2} dx dy = \frac{5}{2} \int_0^2 \left[\int_0^{4-2x} dy \right] dx = \frac{5}{2} \int_0^2 [y]_0^{4-2x} dx = \\ &\sqrt{1 + 9 + \frac{9}{4}} = \sqrt{\frac{49}{4}} = \frac{7}{2} \end{aligned}$$

$$= \frac{5}{2} \int_0^2 (4 - 2x) dx = \frac{5}{2} [4x - x^2]_0^2 = \frac{5}{2} (8 - 4) = \frac{5}{2} \cdot 4 = \underline{10}$$

2) $S: 2z = x^2, y \leq 2x, y \geq \frac{1}{2}x, x \leq 2\sqrt{2}$

$$\begin{aligned} &\rightarrow f: z = \frac{1}{2}x^2 \\ &f'_x = x, f'_y = 0 \end{aligned}$$

$\rightarrow$ Parabolická raševá plocha s řídicí křivkou $2z = x^2$ ležící v rovině (x, z) a tvořícími přímkami rovnoběžnými s osou y .



A :

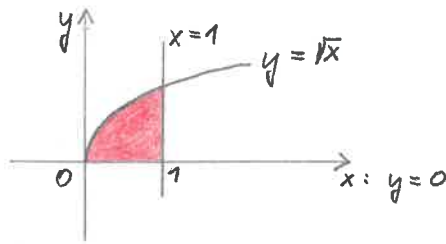
Oblast I. druhu

$$\begin{aligned} &0 \leq x \leq 2\sqrt{2} \\ &\frac{1}{2}x \leq y \leq 2x \end{aligned}$$

$$\begin{aligned} S &= \iint_A \sqrt{1 + x^2 + 0^2} dx dy = \int_0^{2\sqrt{2}} \left[\int_{\frac{1}{2}x}^{2x} \sqrt{1 + x^2} dy \right] dx = \int_0^{2\sqrt{2}} \sqrt{1 + x^2} [y]_{\frac{1}{2}x}^{2x} dx = \\ &= \int_0^{2\sqrt{2}} \sqrt{1 + x^2} \cdot (2x - \frac{1}{2}x) dx = \frac{3}{2} \int_0^{2\sqrt{2}} x \sqrt{1 + x^2} dx = \left| \begin{array}{l} t = 1 + x^2 \\ dt = 2x dx \end{array} \right| \frac{x}{1} \bigg|_0^{2\sqrt{2}} = \\ &= \frac{3}{2} \cdot \frac{1}{2} \int_1^9 \sqrt{t} dt = \frac{3}{4} \left[\frac{2}{3} t^{3/2} \right]_1^9 = \frac{1}{2} [t\sqrt{t}]_1^9 = \frac{1}{2} (9\sqrt{9} - 1\sqrt{1}) = \\ &= \frac{1}{2} (27 - 1) = \underline{13} \end{aligned}$$

Př: Určete těžiště tenké rovinné desky.

1) A: $x=1, y \geq 0, y=\sqrt{x}; \sigma(x,y) = \frac{1}{x+1}$



Oblast I, i II. druhu

$$\begin{aligned} & \hookrightarrow 0 \leq x \leq 1 \\ & 0 \leq y \leq \sqrt{x} \end{aligned}$$

$$\begin{aligned} m &= \iint_A \sigma(x,y) dx dy = \int_0^1 \left[\int_0^{\sqrt{x}} \frac{1}{x+1} dy \right] dx = \int_0^1 \frac{1}{x+1} \cdot [y]_0^{\sqrt{x}} dx = \int_0^1 \frac{\sqrt{x}}{x+1} dx = \\ &= \left| \begin{array}{l} t = \sqrt{x} \\ t^2 = x \\ 2t dt = dx \end{array} \right| \begin{array}{c|c|c} x & 0 & 1 \\ \hline t & 0 & 1 \end{array} = \int_0^1 \frac{t}{t^2+1} \cdot 2t dt = 2 \int_0^1 \frac{t^2+1-1}{t^2+1} dt = \\ & \quad \frac{t^2+1}{t^2+1} - \frac{1}{t^2+1} = 1 - \frac{1}{t^2+1} \\ &= 2 \int_0^1 \left(1 - \frac{1}{t^2+1} \right) dt = 2 \left[t - \arctan t \right]_0^1 = 2 \left(1 - \underbrace{\arctan 1}_{\frac{\pi}{4}} - 0 + \underbrace{\arctan 0}_0 \right) = \\ &= 2 \left(1 - \frac{\pi}{4} \right) = \underline{\underline{\frac{1}{2}(4-\pi)}} \end{aligned}$$

$$\begin{aligned} S_y &= \iint_A x \cdot \sigma(x,y) dx dy = \int_0^1 \left[\int_0^{\sqrt{x}} \frac{x}{x+1} dy \right] dx = \int_0^1 \frac{x}{x+1} \cdot [y]_0^{\sqrt{x}} dx = \int_0^1 \frac{x\sqrt{x}}{x+1} dx = \\ &= \int_0^1 \frac{t^3 \cdot t}{t^2+1} \cdot 2t dt = 2 \int_0^1 \frac{t^4-1+1}{t^2+1} dt = 2 \int_0^1 \left(\frac{(t^2+1)(t^2-1)}{t^2+1} + \frac{1}{t^2+1} \right) dt = \\ &= \int_0^1 (t^2-1 + \frac{1}{t^2+1}) dt = 2 \left[\frac{t^3}{3} - t + \arctan t \right]_0^1 = 2 \left(\frac{1}{3} - 1 + \frac{\pi}{4} \right) = \\ &= 2 \left(\frac{\pi}{4} - \frac{2}{3} \right) = \underline{\underline{\frac{1}{6}(3\pi-8)}} \end{aligned}$$

$$\begin{aligned} S_x &= \iint_A y \cdot \sigma(x,y) dx dy = \int_0^1 \left[\int_0^{\sqrt{x}} \frac{y}{x+1} dy \right] dx = \int_0^1 \frac{1}{x+1} \cdot \left[\frac{y^2}{2} \right]_0^{\sqrt{x}} dx = \frac{1}{2} \int_0^1 \frac{x+1-1}{x+1} dx = \\ &= \frac{1}{2} \int_0^1 \left(1 - \frac{1}{x+1} \right) dx = \frac{1}{2} \left[x - \ln|x+1| \right]_0^1 = \underline{\underline{\frac{1}{2}(1-\ln 2)}} \end{aligned}$$

$$x_T = \frac{S_y}{m} = \frac{\frac{1}{6}(3\pi-8)}{\frac{1}{2}(4-\pi)} = \frac{3\pi-8}{12-3\pi}$$

$$y_T = \frac{S_x}{m} = \frac{\frac{1}{2}(1-\ln 2)}{\frac{1}{2}(4-\pi)} = \frac{1-\ln 2}{4-\pi}$$

$$\underline{\underline{T = \left[\frac{3\pi-8}{12-3\pi}; \frac{1-\ln 2}{4-\pi} \right]}}$$