

Neurčitý integrál

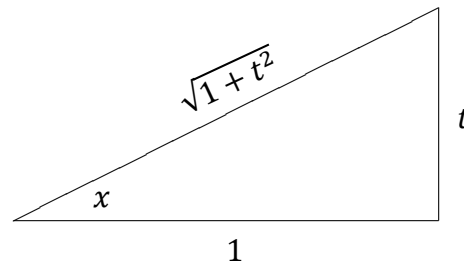
1. Najděte neurčitý integrál $\int \frac{1}{1 + \sin^2 x} dx$ $D(f) = \mathbb{R}$

1.1. Substitute $\operatorname{tg} x = t$

V pravoúhlém trojúhelníku pro úhel x je: $\operatorname{tg} x = \frac{t}{1}$

Tím máme stanoveny velikosti obou odvěsen.

Velikost přepony určíme podle Pythagorovy věty.



$$\begin{aligned} \int \frac{1}{1 + \sin^2 x} dx &= \left| \begin{array}{l} \operatorname{tg} x = t \\ \sin x = \frac{t}{\sqrt{1+t^2}} \end{array} \quad \begin{array}{l} x = \operatorname{arctg} t \\ dx = \frac{1}{1+t^2} dt \end{array} \right| = \int \frac{1}{1 + \frac{t^2}{1+t^2}} \cdot \frac{1}{1+t^2} dt = \\ &= \int \frac{1}{\frac{(1+t^2) + t^2}{1+t^2}} \cdot \frac{1}{1+t^2} dt = \int \frac{1+t^2}{(1+t^2) + t^2} \cdot \frac{1}{1+t^2} dt = \int \frac{1}{1+2t^2} dt = \end{aligned}$$

1.2. Substitute $t = \frac{s}{\sqrt{2}}$

$$\begin{aligned} &= \int \frac{1}{1+2t^2} dt = \left| \begin{array}{l} t = \frac{s}{\sqrt{2}} = \varphi(s) \\ s = \sqrt{2}t = \varphi^{-1}(s) \\ dt = \frac{1}{\sqrt{2}} ds \end{array} \right| = \int \frac{1}{1+2 \cdot \left(\frac{s}{\sqrt{2}}\right)^2} \cdot \frac{1}{\sqrt{2}} ds = \frac{1}{\sqrt{2}} \int \frac{1}{1+s^2} ds = \\ &= \frac{1}{\sqrt{2}} \operatorname{arctg} s = \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2}t) = \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2} \operatorname{tg} x) + \text{konst.} \end{aligned}$$

Tedy $\int \frac{1}{1 + \sin^2 x} dx = \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2} \operatorname{tg} x) + \text{konst.}$

¹ Pokud $\operatorname{tg} x = t$ a požadujeme vyjádřit x (v závislosti na t), tak funkce $\operatorname{tg} x$ není prostá na celém svém definičním oboru. Proto k ní neexistuje inverzní funkce. Pro každý „kousek“ je inverzní funkce jiná (viz pravý sloupec), ale derivace $\frac{dx}{dt} = \frac{1}{1+t^2}$ je stejná pro libovolné x ($\forall x$).

$x \in \left(-\frac{3\pi}{2}; -\frac{\pi}{2}\right)$: $x = \operatorname{arctg} t - \pi$
 $x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$: $x = \operatorname{arctg} t$
 $x \in \left(\frac{\pi}{2}; \frac{3\pi}{2}\right)$: $x = \operatorname{arctg} t + \pi$
 atd.

2. Najděte neurčitý integrál $\int \sin^2 x \, dx$ $D(f) = \mathbb{R}$

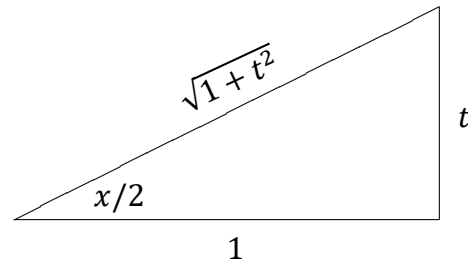
Zde je několik způsobů řešení:

2.1. Substitute $\operatorname{tg} \frac{x}{2} = t$

V pravoúhlém trojúhelníku pro úhel $\frac{x}{2}$ je: $\operatorname{tg} \frac{x}{2} = \frac{t}{1}$

Tím máme stanoveny velikosti obou odvěsen.

Velikost přepony určíme podle Pythagorovy věty.



$$\int \sin^2 x \, dx = \left| \begin{array}{ll} \operatorname{tg} \frac{x}{2} = t & x = 2 \operatorname{arctg} t \\ \sin x = \frac{2t}{1+t^2} & dx = \frac{2}{1+t^2} dt \\ \sin 2\alpha = 2 \sin \alpha \cos \alpha & \sin x = \sin \left(2 \frac{x}{2} \right) \end{array} \right| = \int \left(\frac{2t}{1+t^2} \right)^2 \cdot \frac{2}{1+t^2} dt =$$

$$= \int \frac{8t^2}{(1+t^2)^3} dt = \underbrace{\int \frac{8}{(1+t^2)^2} dt}_{\mathcal{A}} + \int \frac{-8}{(1+t^2)^3} dt = \operatorname{arctg} t + \frac{4t}{1+t^2} - \frac{3t^3+5t}{(1+t^2)^2} + K$$

Vrátit původní proměnnou!

2.1.1. $\mathcal{A} = \int \frac{8}{(1+t^2)^2} dt = 4 \operatorname{arctg} t + \frac{4t}{1+t^2} + \text{konst.}$

$$\int \frac{8}{1+t^2} dt = \left| \begin{array}{ll} u = \frac{1}{1+t^2} & v' = 8 \\ u' = \frac{-2t}{(1+t^2)^2} & v = 8t \end{array} \right| = \frac{8t}{1+t^2} - \int \frac{-16t^2}{(1+t^2)^2} dt$$

Po rozkladu na parciální zlomky $\Rightarrow \frac{8t}{1+t^2} - \int \left[\frac{16}{(1+t^2)^2} - \frac{16}{1+t^2} \right] dt$

$$\int \frac{8}{1+t^2} dt = \frac{8t}{1+t^2} - 2 \int \frac{8}{(1+t^2)^2} dt + \int \frac{16}{1+t^2} dt$$

$$8 \int \frac{1}{1+t^2} dt = \frac{8t}{1+t^2} - 2\mathcal{A} + 16 \int \frac{1}{1+t^2} dt$$

$$2\mathcal{A} = \frac{8t}{1+t^2} + 8 \int \frac{1}{1+t^2} dt$$

$$\mathcal{A} = \dots$$

2.1.2. Analogicky

$$\int \frac{-8}{(1+t^2)^3} dt = -3 \operatorname{arctg} t - \frac{24 t^3}{8 t^4 + 16 t^2 + 8} - \frac{4 t}{8 t^4 + 16 t^2 + 8} + \text{konst.}$$

2.2. Substitute $\operatorname{tg} x = t$ (viz první příklad)

$$\begin{aligned} \int \sin^2 x \, dx &= \left| \begin{array}{ll} \operatorname{tg} x = t & x = \operatorname{arctg} t \\ \sin x = \frac{t}{\sqrt{1+t^2}} & dx = \frac{1}{1+t^2} dt \end{array} \right| = \int \frac{t^2}{1+t^2} \cdot \frac{1}{1+t^2} dt = \int \frac{t^2}{(1+t^2)^2} dt = \\ &= \int \frac{(t^2+1)-1}{(1+t^2)^2} dt = \int \frac{(t^2+1)}{(1+t^2)^2} dt - \int \frac{1}{(1+t^2)^2} dt = \int \frac{1}{1+t^2} - \left\{ \begin{array}{l} \text{výpočet integrálu } \mathcal{A} \\ \text{na předchozí straně} \end{array} \right\} = \\ &= \frac{1}{2} \operatorname{arctg} t - \frac{t}{2(1+t^2)} + K = \frac{1}{2} \operatorname{arctg} (\operatorname{tg} x) - \frac{\operatorname{tg} x}{2(1+\operatorname{tg}^2 x)} + K \end{aligned}$$

2.3. Per partes

$$\begin{aligned} \int \sin^2 x \, dx &= \left| \begin{array}{ll} u = \sin x & v' = \sin x \\ u' = \cos x & v = -\cos x \end{array} \right| = \sin x(-\cos x) - \int \cos x(-\cos x) \, dx = \\ &= -\sin x \cos x + \int \cos^2 x \, dx = -\sin x \cos x + \int (1 - \sin^2 x) \, dx \\ \int \sin^2 x \, dx &= -\sin x \cos x + \int 1 \, dx - \int \sin^2 x \, dx \\ 2 \int \sin^2 x \, dx &= -\sin x \cos x + x + \text{konst.} \\ \int \sin^2 x \, dx &= -\frac{1}{2} \sin x \cos x + \frac{1}{2} x + K \end{aligned}$$

2.4. Vzorec $\sin^2 x = \frac{1 - \cos 2x}{2}$

$$\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx = \frac{1}{2} \int dx - \frac{1}{2} \int \cos 2x \, dx = \frac{x}{2} - \frac{1}{2} \int \cos 2x \, dx =$$

$$\begin{aligned}
 &= \left| \begin{array}{l} 2x = t \\ x = \frac{t}{2} \\ dx = \frac{1}{2} dt \end{array} \right| = \frac{x}{2} - \frac{1}{2} \int \cos\left(2\frac{t}{2}\right) \frac{1}{2} dt = \frac{x}{2} - \frac{1}{4} \int \cos t dt = \frac{x}{2} - \frac{1}{4} \sin t + \text{konst.} = \\
 &= \frac{x}{2} - \frac{1}{4} \sin 2x + K
 \end{aligned}$$

2.4.1. Odvození vzorce, když

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$1 = \sin^2 x + \cos^2 x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = (1 - \sin^2 x) - \sin^2 x$$

$$\cos 2x = 1 - 2 \sin^2 x$$

$$2 \sin^2 x = 1 - \cos 2x$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$