

Neurčitý integrál

$$\begin{aligned}
 1. \quad [x \in \mathbb{R} \setminus \{0\}] \quad & \int \frac{3x^4 + 5x^3 - 6x \cdot \sqrt[3]{x} + 4}{x} dx = \int \left(3x^3 + 5x^2 - 6x^{\frac{1}{3}} + \frac{4}{x} \right) dx = \\
 & = 3 \frac{x^{3+1}}{4} + 5 \frac{x^{2+1}}{3} - 6 \frac{x^{\frac{1}{3}+1}}{\frac{4}{3}} + 4 \ln |x| + c = \underline{\underline{\frac{3}{4}x^4 + \frac{5}{3}x^3 - \frac{9}{2} \cdot \sqrt[3]{x^4} + \ln x^4 + c}}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad [x \in \mathbb{R} \setminus \{0; -1\}] \quad & \int \frac{x^2 - 3x + 2}{x^3 + 2x^2 + x} dx = \int \left(\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \right) dx = \\
 & \quad \begin{aligned} x^2 - 3x + 2 &= A(x+1)^2 + Bx(x+1) + Cx \\ x=0: \quad & 2 = A \\ x=-1: \quad & 6 = -C \quad C = -6 \\ x^2: \quad & 1 = A + B \quad B = -1 \end{aligned} \\
 & = 2 \int \frac{1}{x} dx - \int \frac{1}{x+1} dx - 6 \int \frac{1}{(x+1)^2} dx = \left| \begin{array}{l} y = x+1 \\ y' = 1 \\ \frac{dy}{dx} = 1 \\ dy = dx \end{array} \quad \frac{1}{y^2} = y^{-2} \right| = \\
 & = 2 \ln |x| - \ln |x+1| - 6 \int y^{-2} dy = \ln \frac{x^2}{|x+1|} - 6 \cdot \frac{y^{-1}}{-1} + c = \underline{\underline{\ln \frac{x^2}{|x+1|} + \frac{6}{x+1} + c}}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad [x \in \mathbb{R} \setminus \{-1\}] \quad & \int \frac{x-1}{x+1} dx = \int \frac{(x+1)-2}{x+1} dx = \int \left(\frac{x+1}{x+1} - \frac{2}{x+1} \right) dx = \\
 & = \int 1 dx - 2 \int \frac{1}{x+1} dx = \left| \int \frac{f'(x)}{f(x)} dx \right| = x - 2 \ln |x+1| + c = \underline{\underline{x - \ln(x+1)^2 + c}}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad [x \in \mathbb{R}] \quad & \int \frac{x^2-1}{x^2+1} dx = \int \frac{(x^2+1)-2}{x^2+1} dx = \int \left(\frac{x^2+1}{x^2+1} - \frac{2}{x^2+1} \right) dx = \\
 & = \int 1 dx - \int \frac{2}{x^2+1} dx = \underline{\underline{x - 2 \operatorname{arctg} x + c}}
 \end{aligned}$$

$$\begin{aligned}
 4.a \quad [x \in \mathbb{R}] \quad & \int \frac{1}{x^2+2x+2} dx = \int \frac{1}{(x^2+2x+1)+1} dx = \int \frac{1}{(x+1)^2+1} dx = \\
 & = \left| \begin{array}{l} z = x+1 \\ dz = dx \end{array} \right| = \int \frac{1}{z^2+1} dz = \operatorname{arctg} z + c = \underline{\underline{\operatorname{arctg}(x+1) + c}}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad [x \in \mathbb{R}] \quad \int \frac{x^4}{x^2+1} dx &= \int \frac{(x^4+x^2)-x^2}{x^2+1} dx = \int \left[\frac{x^2(x^2+1)}{x^2+1} - \frac{(x^2+1)-1}{x^2+1} \right] dx = \\
 &= \int x^2 dx - \int 1 dx - \int \frac{-1}{x^2+1} dx = \underline{\underline{\frac{1}{3}x^3 - x + \operatorname{arctg} x + c}}
 \end{aligned}$$

$$6. \quad [x \in \mathbb{R} \setminus \{1\}] \quad \int \frac{x}{x^3-1} dx = \int \frac{x}{(x-1)(x^2+x+1)} dx =$$

$$= \int \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \right) dx =$$

$$x = A(x^2+x+1) + (Bx+C)(x-1)$$

$$\begin{array}{lll}
 x = 1 : & 1 = 3A & A = \frac{1}{3} \\
 x^2 : & 0 = A + B & B = -\frac{1}{3} \\
 x^0 : & 0 = A - C & C = \frac{1}{3}
 \end{array}$$

$$= \int \left(\frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}x + \frac{1}{3}}{x^2+x+1} \right) dx = \frac{1}{3} \int \frac{1}{\underbrace{x-1}_{\frac{f'(x)}{f(x)}}} dx - \frac{1}{3} \int \frac{x-1}{x^2+x+1} dx =$$

$$= \frac{1}{3} \ln|x-1| - \frac{1}{3} \int \frac{\frac{1}{2}(2x+1) - \frac{3}{2}}{x^2+x+1} dx =$$

$$= \frac{1}{3} \ln|x-1| - \frac{1}{6} \int \frac{2x+1}{\underbrace{x^2+x+1}_{\frac{f'(x)}{f(x)}}} dx + \frac{1}{2} \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} dx = \left| \begin{array}{l} x + \frac{1}{2} = \sqrt{\frac{3}{4}} \cdot u \\ dx = \frac{\sqrt{3}}{2} du \\ \frac{2x+1}{\sqrt{3}} = u \end{array} \right| =$$

$$= \frac{1}{6} \left(2 \ln|x-1| - \ln \underbrace{|x^2+x+1|}_{\text{kompl. k.}} \right) + \frac{1}{2} \int \frac{1}{\frac{3}{4}u^2 + \frac{3}{4}} \cdot \frac{\sqrt{3}}{2} du =$$

$$= \frac{1}{6} \ln \frac{(x-1)^2}{x^2+x+1} + \frac{1}{2} \cdot \frac{1}{\frac{3}{4}} \cdot \frac{\sqrt{3}}{2} \int \frac{1}{u^2+1} du = \frac{1}{6} \ln \frac{(x-1)^2}{x^2+x+1} + \frac{1}{\sqrt{3}} \operatorname{arctg} u + c =$$

$$= \underline{\underline{\frac{1}{6} \ln \frac{(x-1)^2}{x^2+x+1} + \frac{\sqrt{3}}{3} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} + c}}$$

$$\begin{aligned}
 7. \quad [x \in (-2; 2)] \quad \int \frac{x}{\sqrt{4-x^2}} dx &= \left| \begin{array}{l} 4-x^2 = s \\ -2x dx = ds \\ dx = \frac{-1}{2x} ds \end{array} \right| = \int \frac{x}{\sqrt{s}} \cdot \frac{-1}{2x} ds = \\
 &= -\frac{1}{2} \int s^{-\frac{1}{2}} ds = -\frac{1}{2} \cdot \frac{s^{\frac{1}{2}}}{\frac{1}{2}} + c = -\sqrt{s} + c = \underline{\underline{-\sqrt{4-x^2} + c}}
 \end{aligned}$$

$$\begin{aligned}
 8. \quad [x \in (0; \infty)] \quad \int \frac{\ln^2 x}{x} dx &= \left| \begin{array}{l} \ln x = v \\ \frac{1}{x} dx = dv \\ dx = x dv \end{array} \right| = \int \frac{v^2}{x} \cdot x dv = \int v^2 dv = \frac{1}{3} v^3 + c = \\
 &= \underline{\underline{\frac{1}{3} \ln^3 x + c}}
 \end{aligned}$$

$$\begin{aligned}
 9. \quad [x \in \mathbb{R}] \quad \int \frac{\sqrt[3]{\arctg x}}{1+x^2} dx &= \left| \begin{array}{l} \arctg x = v \\ \frac{1}{1+x^2} dx = dv \\ dx = (1+x^2) dv \end{array} \right| = \int \frac{\sqrt[3]{v}}{1+x^2} \cdot (1+x^2) dv = \\
 &= \int \sqrt[3]{v} dv = \int v^{\frac{1}{3}} dv = \frac{v^{\frac{4}{3}}}{\frac{4}{3}} + c = \underline{\underline{\frac{3}{4} \cdot \sqrt[3]{\arctg^4 x} + c}}
 \end{aligned}$$

$$\begin{aligned}
 10. \quad [x \in (0; \infty)] \quad \int \frac{\cos(\ln x)}{x} dx &= \left| \begin{array}{l} \ln x = w \\ \frac{1}{x} dx = dw \\ dx = x dw \end{array} \right| = \int \frac{\cos w}{x} \cdot x dw = \int \cos w dw = \\
 &= \sin w + c = \underline{\underline{\sin(\ln x) + c}}
 \end{aligned}$$

$$\begin{aligned}
 11. \quad [x \neq (2k+1)\frac{\pi}{2}] \quad \int \frac{e^{\tg x}}{\cos^2 x} dx &= \left| \begin{array}{l} \tg x = y \\ \frac{1}{\cos^2 x} dx = dy \\ dx = \cos^2 x dy \end{array} \right| = \int \frac{e^y}{\cos^2 x} \cdot \cos^2 x dy = \\
 &= \int e^y dy = e^y + c = \underline{\underline{e^{\tg x} + c}}
 \end{aligned}$$

$$\begin{aligned}
 \text{12. } [x \in (0; \infty)] \quad \int \frac{\ln^2 x + 1}{x} dx &= \left| \begin{array}{l} \ln x = v \\ \frac{1}{x} dx = dv \\ dx = x dv \end{array} \right| = \int \frac{v^2 + 1}{x} \cdot x dv = \\
 &= \int (v^2 + 1) dv = \frac{1}{3} v^3 + v + c = \underline{\underline{\frac{1}{3} \ln^3 x + \ln x + c}}
 \end{aligned}$$

$$\begin{aligned}
 \text{13. } [x \in \mathbb{R}] \quad \int \frac{2x - 1}{\sqrt{1 - x + x^2}} dx &= \left| \begin{array}{l} 1 - x + x^2 = u \\ (-1 + 2x) dx = du \\ dx = \frac{1}{2x-1} du \end{array} \right| = \\
 &= \int \frac{2x - 1}{\sqrt{u}} \cdot \frac{1}{2x - 1} du = \int u^{-\frac{1}{2}} du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} = \underline{\underline{2\sqrt{1 - x + x^2} + c}}
 \end{aligned}$$

$$\begin{aligned}
 \text{14. } [x \in \mathbb{R}] \quad \int (x^2 + 1)^3 \cdot 2x dx &= \left| \begin{array}{l} x^2 + 1 = t \\ 2x dx = dt \\ dx = \frac{1}{2x} dt \end{array} \right| = \int t^3 \cdot 2x \cdot \frac{1}{2x} dt = \int t^3 dt = \\
 &= \frac{t^4}{4} + c = \underline{\underline{\frac{1}{4}(x^2 + 1)^4 + c}}
 \end{aligned}$$

$$\begin{aligned}
 \text{15. } \left[x \in \left(0; \frac{1}{e}\right) \cup \left(\frac{1}{e}; \infty\right) \right] \quad \int \frac{1}{x \cdot (1 + \ln x)} dx &= \left| \begin{array}{l} 1 + \ln x = z \\ \frac{1}{x} dx = dz \\ dx = x dz \end{array} \right| = \\
 &= \int \frac{1}{x \cdot z} \cdot x dz = \int \frac{1}{z} dz = \ln |z| + c = \underline{\underline{\ln |1 + \ln x| + c}}
 \end{aligned}$$

$$\begin{aligned}
 \text{16. } [x \in (1; 3)] \quad \int \frac{1}{\sqrt{4x - 3 - x^2}} dx &= \int \frac{1}{\sqrt{-3 - (x^2 - 4x)}} dx = \\
 &= \int \frac{1}{\sqrt{-3 - (x^2 - 4x + 4 - 4)}} dx = \int \frac{1}{\sqrt{-3 - (x^2 - 4x + 4) + 4}} dx = \\
 &= \int \frac{1}{\sqrt{1 - (x - 2)^2}} dx = \left| \begin{array}{l} x - 2 = y \\ dx = dy \end{array} \right| = \int \frac{1}{\sqrt{1 - y^2}} dy = \arcsin y + c = \\
 &= \underline{\underline{\arcsin(x - 2) + c}}
 \end{aligned}$$

$$\begin{aligned} 17. \quad [x \in \mathbb{R} \setminus \{\pm 2\}] \quad & \int \frac{2x^2 - 8}{16 - x^4} \, dx = \int \frac{2(x^2 - 4)}{(4 - x^2)(4 + x^2)} \, dx = -2 \int \frac{1}{4 + x^2} \, dx = \\ & = -2 \int \frac{1}{4\left(1 + \frac{x^2}{4}\right)} \, dx = -\frac{1}{2} \int \frac{1}{1 + \left(\frac{x}{2}\right)^2} \, dx = \left| \begin{array}{l} \frac{x}{2} = u \\ x = 2u \\ dx = 2 \, du \end{array} \right| = \\ & = -\frac{1}{2} \int \frac{1}{1 + u^2} \cdot 2 \, du = -\operatorname{arctg} u + c = \underline{\underline{-\operatorname{arctg} \frac{x}{2} + c}} \end{aligned}$$

$$\begin{aligned}
18. \quad [x \in (0; 1) \cup (1; \infty)] \quad & \int \frac{\sqrt{x}}{\sqrt[3]{x^2} - \sqrt[4]{x}} dx = \int \frac{x^{\frac{1}{2}}}{x^{\frac{2}{3}} - x^{\frac{1}{4}}} dx = \\
& = \left| \frac{1}{2}; \frac{2}{3}; \frac{1}{4} \Rightarrow \text{společný jmenovatel je } 12 \Rightarrow \begin{array}{l} x = z^{12} \\ dx = 12z^{11} dz \\ \sqrt[12]{x} = z \end{array} \right| = \\
& = \int \frac{(z^{12})^{\frac{1}{2}}}{(z^{12})^{\frac{2}{3}} - (z^{12})^{\frac{1}{4}}} 12z^{11} dz = 12 \int \frac{z^6}{z^8 - z^3} z^{11} dz = 12 \int \frac{z^{17}}{z^3(z^5 - 1)} dz = \\
& = 12 \int \frac{z^{14}}{z^5 - 1} dz = 12 \int \frac{z^{10}}{z^5 - 1} z^4 dz = \left| \begin{array}{l} z^5 - 1 = t \\ z^5 = t + 1 \\ 5z^4 dz = dt \\ z^4 dz = \frac{1}{5} dt \\ z^{10} = (z^5)^2 \end{array} \right| = 12 \int \frac{(t+1)^2}{t} \cdot \frac{1}{5} dt = \\
& = \frac{12}{5} \int \frac{t^2 + 2t + 1}{t} dt = \frac{12}{5} \int \left(t + 2 + \frac{1}{t} \right) dt = \frac{12}{5} \left(\frac{t^2}{2} + 2t + \ln |t| \right) + c = \\
& = \frac{12}{10} [(z^5 - 1)^2 + 4(z^5 - 1) + 2 \ln |(z^5 - 1)|] + c = \\
& = \frac{12}{10} \left[\left(\sqrt[12]{x^5} - 1 \right)^2 + 4 \left(\sqrt[12]{x^5} - 1 \right) + \ln \left(\sqrt[12]{x^5} - 1 \right)^2 \right] + c
\end{aligned}$$

$$\begin{aligned}
19. \quad [x \in (-2; 2)] \quad \int \frac{x^2}{\sqrt{4-x^2}} dx &= \left| \begin{array}{l} x = 2 \sin y \\ dx = 2 \cos y dy \\ \frac{x}{2} = \sin y \\ \arcsin \frac{x}{2} = y \end{array} \right| = \\
&= \int \frac{(2 \sin y)^2}{\sqrt{4 - (2 \sin y)^2}} 2 \cos y dy = \int \frac{4 \sin^2 y}{\sqrt{4 - 4 \sin^2 y}} 2 \cos y dy = \\
&= \int \frac{8 \sin^2 y \cdot \cos y}{\sqrt{4} \cdot \sqrt{1 - \sin^2 y}} dy = \int \frac{8 \sin^2 y \cdot \cos y}{2 \sqrt{\cos^2 y}} dy = 4 \int \frac{\sin^2 y \cdot \cos y}{|\cos y|} dy = \\
&= \left| \begin{array}{l} \arcsin \frac{x}{2} = y \\ H\left(\arcsin \frac{x}{2}\right) = \left(-\frac{\pi}{2}; \frac{\pi}{2}\right) \\ \left(-\frac{\pi}{2}; \frac{\pi}{2}\right) = D(y) \\ \cos y > 0 \end{array} \right| = 4 \int \frac{\sin^2 y \cdot \cos y}{\cos y} dy = 4 \int \sin^2 y dy = \\
&= 4 \int \frac{1 - \cos 2y}{2} dy = 2 \int dy - \int \cos 2y \cdot 2 dy = \left| \begin{array}{l} 2y = z \\ 2 dy = dz \end{array} \right| = \\
&= 2y - \int \cos z dz = 2 \arcsin \frac{x}{2} - \sin z + c = 2 \arcsin \frac{x}{2} - \sin 2y + c = \\
&= 2 \arcsin \frac{x}{2} - 2 \sin y \cdot \cos y + c = 2 \arcsin \frac{x}{2} - x \sqrt{1 - \sin^2 y} + c = \\
&= 2 \arcsin \frac{x}{2} - x \sqrt{1 - \left(\frac{x}{2}\right)^2} + c = 2 \arcsin \frac{x}{2} - x \sqrt{\frac{4 - x^2}{4}} + c = \\
&= \underline{\underline{2 \arcsin \frac{x}{2} - \frac{x}{2} \sqrt{4 - x^2} + c}}
\end{aligned}$$