Neurčitý integrál

1. $[x \in \mathbb{R}]$

$$\underline{\int x \cdot e^{2x}\,dx} = \left| \begin{array}{ll} u = x & v' = e^{2x} \\ u' = 1 & v = \frac{1}{2}e^{2x} \end{array} \right| = \frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}\,dx = \underline{\underline{\frac{1}{4}e^{2x}(2x-1) + c}}$$

2. $[x \in \mathbb{R}]$

$$\underline{\int x \cdot \operatorname{arctg} x\,dx} = \left| \begin{array}{ll} u = \operatorname{arctg} x & v' = x \\ u' = \frac{1}{1+x^2} & v = \frac{x^2}{2} \end{array} \right| =$$

$$= \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx = \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int \frac{(x^2+1)-1}{1+x^2}\,dx =$$

$$\underline{\underline{\frac{x^2+1}{2}\operatorname{arctg} x - \frac{x}{2} + c}}$$

3. $[x \in \mathbb{R} \setminus \{0; -1\}]$

$$\underline{\int \frac{x^2-3x+2}{x^3+2x^2+x}\,dx} = \int \left(\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}\right) dx =$$

$$x^2 - 3x + 2 = A\,(x+1)^2 + B\,x(x+1) + C\,x$$

$$\begin{array}{rll} x = 0: & 2 = A & \\ x = -1: & 6 = -C & C = -6 \\ x^2: & 1 = A + B & B = -1 \end{array}$$

$$= 2\int \frac{1}{x}\,dx - \underbrace{\int \frac{1}{x+1}}_{\frac{f'(x)}{f(x)}}\,dx - 6\underbrace{\int \frac{1}{(x+1)^2}\,dx}_{\mathcal{A}\text{(lze i substitucí)}} = \left| \int \frac{1}{(x+1)^2}\,dx = \frac{-1}{x+1} + c_1 \right|$$

$$\mathcal{A}: \int 1 \cdot \frac{1}{x+1}\,dx = \left| \begin{array}{ll} u = \frac{1}{x+1} & v' = 1 \\ u' = \frac{-1}{(x+1)^2} & v = x \end{array} \right| = \frac{1}{x+1} \cdot x - \int \frac{-1}{(x+1)^2} \cdot x\,dx$$

$$\int \frac{1}{x+1}\,dx = \frac{x}{x+1} + \int \frac{x}{(x+1)^2}\,dx = \frac{x}{x+1} + \int \frac{(x+1)-1}{(x+1)^2}\,dx$$

$$\int \frac{1}{x+1}\,dx = \frac{x}{x+1} + \int \frac{1}{x+1}\,dx - \int \frac{1}{(x+1)^2}\,dx$$

$$\int \frac{1}{(x+1)^2}\,dx = \frac{x}{x+1} + c_2 = \frac{x}{x+1} \underbrace{-1 + c_1}_{c_2} = \frac{x-x-1}{x+1} + c_1 = \frac{-1}{x+1} + c_1$$

$$= 2\ln|x| - \ln|x+1| - 6 \cdot \left(\frac{-1}{x+1} + c_1\right) = \ln x^2 - \ln|x+1| + \frac{6}{x+1} \underbrace{-6c_1}_{c} =$$

$$= \underline{\underline{\ln\frac{x^2}{|x+1|} + \frac{6}{x+1} + c}}$$

4. [$x \in (0\,;\,\infty)$] $\displaystyle \underline{\int \frac{\ln x}{x^2}\,\mathrm{d}x} = \left| \begin{matrix} u = \ln x & v' = \frac{1}{x^2} \\ u' = \frac{1}{x} & v = -\frac{1}{x} \end{matrix} \right| = -\frac{\ln x}{x} - \int -\frac{1}{x^2}\,\mathrm{d}x =$

$$= -\frac{\ln x}{x} + \int x^{-2}\,\mathrm{d}x = -\frac{\ln x}{x} + \frac{x^{-1}}{-1} + c = \underline{\underline{-\frac{1}{x}\cdot(1+\ln x) + c}}$$

5. [$x \in \langle -1\,;\,1\rangle$] $\displaystyle \underline{\int \arcsin x\,\mathrm{d}x} = \left| \begin{matrix} u = \arcsin x & v' = 1 \\ u' = \frac{1}{\sqrt{1-x^2}} & v = x \end{matrix} \right| =$

$$= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = x\arcsin x + \int \frac{-x}{\sqrt{1-x^2}}\,\mathrm{d}x = \left| \begin{matrix} w = \sqrt{1-x^2} \\ w^2 = 1-x^2 \\ 2w\,\mathrm{d}w = -2x\,\mathrm{d}x \\ \frac{w}{-x}\,\mathrm{d}w = \mathrm{d}x \end{matrix} \right| =$$

$$= x\arcsin x + \int \frac{-x}{w}\cdot\frac{w}{-x}\,\mathrm{d}w = x\arcsin x + \int \mathrm{d}w = x\arcsin x + w + c =$$

$$= \underline{\underline{x\arcsin x + \sqrt{1-x^2} + c}}$$

6. [$x \in \mathbb{R}$] $\displaystyle \underline{\int (4x+5)\cdot 2^{3x-1}\,\mathrm{d}x} = \int (4x+5)\cdot \mathrm{e}^{(3x-1)\ln 2}\,\mathrm{d}x$

$$= \left| \begin{matrix} u = 4x+5 & v' = \mathrm{e}^{(3x-1)\ln 2} \\ u' = 4 & v = \frac{\mathrm{e}^{(3x-1)\ln 2}}{3\ln 2} \end{matrix} \right| = (4x+5)\cdot\frac{\mathrm{e}^{(3x-1)\ln 2}}{3\ln 2} - 4\int \frac{\mathrm{e}^{(3x-1)\ln 2}}{3\ln 2}\,\mathrm{d}x =$$

$$= \underline{\underline{\frac{4x+5}{3\ln 2}\cdot 2^{3x-1} - \frac{4}{(3\ln 2)^2}\cdot 2^{3x-1} + c}}$$

7. [$x \in (1\,;\,\infty)$] $\displaystyle \underline{\int x\ln(x-1)\,\mathrm{d}x} = \left| \begin{matrix} u = \ln(x-1) & v' = x \\ u' = \frac{1}{x-1} & v = \frac{x^2}{2} \end{matrix} \right| =$

$$= \frac{x^2}{2}\ln(x-1) - \frac{1}{2}\int \frac{x^2}{x-1}\,\mathrm{d}x = \frac{x^2}{2}\ln(x-1) - \frac{1}{2}\int \frac{[x(x-1)]+x}{x-1}\,\mathrm{d}x =$$

$$= \frac{x^2}{2}\ln(x-1) - \frac{1}{2}\int x\,\mathrm{d}x - \frac{1}{2}\int \frac{(x-1)+1}{x-1}\,\mathrm{d}x =$$

$$= \frac{x^2}{2}\ln(x-1) - \frac{1}{2}\cdot\frac{x^2}{2} - \frac{1}{2}\int \mathrm{d}x - \frac{1}{2}\int \underbrace{\frac{1}{x-1}}_{\frac{f'(x)}{f(x)}}\,\mathrm{d}x =$$

$$= \frac{x^2}{2}\ln(x-1) - \frac{x^2}{4} - \frac{x}{2} - \frac{1}{2}\ln(x-1) + c = \underline{\underline{-\frac{x^2+2x}{4} + \frac{x^2-1}{2}\ln(x-1) + c}}$$

8. $[x \in \mathbb{R}]$

$$\underline{\int x \cdot \mathrm{e}^{-x}\,\mathrm{d}x} = \left| \begin{array}{ll} u = x & v' = \mathrm{e}^{-x} \\ u' = 1 & v = -\mathrm{e}^{-x} \\ & y = -x \\ & \mathrm{d}y = -\,\mathrm{d}x \end{array} \right| = -x\mathrm{e}^{-x} - \int -\mathrm{e}^{-x}\,\mathrm{d}x =$$

$$= \underline{\underline{-\frac{x+1}{\mathrm{e}^x} + c}}$$

9. $[x \in \mathbb{R}]$

$$\underline{\int x\cos x\,\mathrm{d}x} = \left| \begin{array}{ll} u = x & v' = \cos x \\ u' = 1 & v = \sin x \end{array} \right| = x\sin x - \int \sin x\,\mathrm{d}x =$$

$$= \underline{\underline{x\sin x + \cos x + c}}$$

10. $[x \in \mathbb{R}]$

$$\underline{\int \mathrm{e}^x \sin x\,\,\mathrm{d}x} = \left| \begin{array}{ll} u_1 = \mathrm{e}^x & v_1' = \sin x \\ u_1' = \mathrm{e}^x & v_1 = -\cos x \end{array} \right| =$$

$$= -\mathrm{e}^x\cos x - \int -\mathrm{e}^x\cos x\,\mathrm{d}x = -\mathrm{e}^x\cos x + \int \mathrm{e}^x\cos x\,\mathrm{d}x = \left| \begin{array}{ll} u_2 = \mathrm{e}^x & v_2' = \cos x \\ u_2' = \mathrm{e}^x & v_2 = \sin x \end{array} \right| =$$

$$\underline{= -\mathrm{e}^x\cos x + \mathrm{e}^x\sin x - \int \mathrm{e}^x\sin x\,\mathrm{d}x} \qquad \Downarrow$$

$$\underline{\int \mathrm{e}^x \sin x\,\,\mathrm{d}x} = -\mathrm{e}^x\cos x + \mathrm{e}^x\sin x - \underline{\int \mathrm{e}^x\sin x\,\mathrm{d}x}$$

$$2\int \mathrm{e}^x \sin x\,\,\mathrm{d}x = -\mathrm{e}^x\cos x + \mathrm{e}^x\sin x$$

$$\int \mathrm{e}^x \sin x\,\,\mathrm{d}x = \underline{\underline{\tfrac{1}{2}\mathrm{e}^x\sin x - \tfrac{1}{2}\mathrm{e}^x\cos x + c}}$$

11. $[x \in \mathbb{R}]$

$$\underline{\int x^2\sin x\,\,\mathrm{d}x} = \left| \begin{array}{ll} u_1 = x^2 & v_1' = \sin x \\ u_1' = 2x & v_1 = -\cos x \end{array} \right| = -x^2\cos x - \int -2x\cos x\,\mathrm{d}x =$$

$$= -x^2\cos x + \int 2x\cos x\,\mathrm{d}x = \left| \begin{array}{ll} u_2 = 2x & v_2' = \cos x \\ u_2' = 2 & v_2 = \sin x \end{array} \right| =$$

$$= -x^2\cos x + 2x\sin x - \int 2\sin x\,\mathrm{d}x = \underline{\underline{-x^2\cos x + 2x\sin x + 2\cos x + c}}$$

12. [$x \neq k\pi$] $$\underline{\int \frac{x}{\sin^2 x}\,\mathrm{d}x} = \left| \begin{matrix} u = x & v' = \frac{1}{\sin^2 x} \\ u' = 1 & v = -\operatorname{cotg} x \end{matrix} \right| =$$

$$= -x \operatorname{cotg} x - \int -\underbrace{\frac{\cos x}{\sin x}}_{\frac{f'(x)}{f(x)}}\,\mathrm{d}x = \underline{\underline{-x \operatorname{cotg} x + \ln|\sin x| + c}}$$

13. [$x \in (-1\,;\,\infty)$] $$\underline{\int \ln(x+1)\,\mathrm{d}x} = \left| \begin{matrix} u = \ln(x+1) & v' = 1 \\ u' = \frac{1}{x+1} & v = x \end{matrix} \right| =$$

$$= x\ln(x+1) - \int \frac{x}{x+1}\,\mathrm{d}x = x\ln(x+1) - \int \frac{(x+1)-1}{x+1}\,\mathrm{d}x =$$

$$= x\ln(x+1) - \int \frac{x+1}{x+1}\,\mathrm{d}x - \int \frac{-1}{x+1}\,\mathrm{d}x = x\ln(x+1) - \int \mathrm{d}x + \int \underbrace{\frac{1}{x+1}}_{\frac{f'(x)}{f(x)}}\,\mathrm{d}x =$$

$$= \underline{\underline{x\ln(x+1) - x + \ln(x+1) + c}}$$

14. [$x > 0$] $$\underline{\int \ln^3 x\,\mathrm{d}x} = \left| \begin{matrix} u_1 = \ln^3 x & v_1' = 1 \\ u_1' = 3\ln^2 x \cdot \left(\frac{1}{x}\right) & v_1 = x \end{matrix} \right| = x\ln^3 x - \int 3\ln^2 x\,\mathrm{d}x =$$

$$= \left| \begin{matrix} u_2 = \ln^2 x & v_2' = 1 \\ u_2' = 2\ln x \cdot \left(\frac{1}{x}\right) & v_2 = x \end{matrix} \right| = x\ln^3 x - 3\left(x\ln^2 x - \int 2\ln x\,\mathrm{d}x\right) =$$

$$= \left| \begin{matrix} u_3 = \ln x & v_3' = 1 \\ u_3' = \frac{1}{x} & v_3 = x \end{matrix} \right| = x\ln^3 x - 3x\ln^2 x + 6\left(x\ln x - \int \mathrm{d}x\right) =$$

$$= \underline{\underline{x\ln^3 x - 3x\ln^2 x + 6x\ln x - 6x + c}}$$