Neurčitý integrál

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1. a) $[x \in (-2\,;\,2)]$

$$\int \frac{x}{\sqrt{4-x^2}}\,\mathrm{d}x = \left| \begin{aligned} 4-x^2 &= s \\ -2x\,\mathrm{d}x &= \mathrm{d}s \\ \mathrm{d}x &= \tfrac{-1}{2x}\,\mathrm{d}s \end{aligned} \right| = \int \frac{x}{\sqrt{s}}\cdot\frac{-1}{2x}\,\mathrm{d}s =$$

$$= -\frac{1}{2}\int s^{-\frac{1}{2}}\,\mathrm{d}s = -\frac{1}{2}\cdot\frac{s^{\frac{1}{2}}}{\frac{1}{2}} + c = -\sqrt{s} + c = \underline{\underline{-\sqrt{4-x^2} + c}}$$

1. b) $[x \in (-2\,;\,2)]$

$$\int \frac{x}{\sqrt{4-x^2}}\,\mathrm{d}x = \left| \begin{aligned} \sqrt{4-x^2} &= t \\ 4-x^2 &= t^2 \\ -2x\,\mathrm{d}x &= 2t\,\mathrm{d}t \\ \mathrm{d}x &= \tfrac{-t}{x}\,\mathrm{d}t \end{aligned} \right| = \int \frac{x}{t}\cdot\frac{-t}{x}\,\mathrm{d}t =$$

$$= -\int \mathrm{d}t = -t + c = \underline{\underline{-\sqrt{4-x^2} + c}}$$

2. a) $[x \in \mathbb{R}]$

$$\int \frac{2x-1}{\sqrt{1-x+x^2}}\,\mathrm{d}x = \left| \begin{aligned} 1-x+x^2 &= u \\ (-1+2x)\,\mathrm{d}x &= \mathrm{d}u \\ \mathrm{d}x &= \tfrac{1}{2x-1}\,\mathrm{d}u \end{aligned} \right| =$$

$$= \int \frac{2x-1}{\sqrt{u}}\cdot\frac{1}{2x-1}\,\mathrm{d}u = \int u^{-\frac{1}{2}}\,\mathrm{d}u = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c = \underline{\underline{2\cdot\sqrt{1-x+x^2} + c}}$$

2. b) $[x \in \mathbb{R}]$

$$\int \frac{2x-1}{\sqrt{1-x+x^2}}\,\mathrm{d}x = \left| \begin{aligned} \sqrt{1-x+x^2} &= v \\ 1-x+x^2 &= v^2 \\ (-1+2x)\,\mathrm{d}x &= 2v\,\mathrm{d}v \\ \mathrm{d}x &= \tfrac{2v}{2x-1}\,\mathrm{d}v \end{aligned} \right| =$$

$$= \int \frac{2x-1}{v}\cdot\frac{2v}{2x-1}\,\mathrm{d}v = 2\int \mathrm{d}v = 2v + c = \underline{\underline{2\cdot\sqrt{1-x+x^2} + c}}$$

3. [$x \in (0\,;\,1) \cup (1\,;\,\infty)$] $\underline{\displaystyle\int \frac{\sqrt{x}}{\sqrt[3]{x^2}-\sqrt[4]{x}}\,\mathrm{d}x} = \displaystyle\int \frac{x^{\frac{1}{2}}}{x^{\frac{2}{3}}-x^{\frac{1}{4}}}\,\mathrm{d}x =$

$$= \left| \frac{1}{2};\ \frac{2}{3};\ \frac{1}{4} \ \Rightarrow \text{nejmenší společný jmenovatel je } 12 \Longrightarrow \begin{array}{r} \sqrt[12]{x} = z \\ x = z^{12} \\ \mathrm{d}x = 12z^{11}\,\mathrm{d}z \end{array} \right| =$$

$$= \int \frac{(z^{12})^{\frac{1}{2}}}{(z^{12})^{\frac{2}{3}}-(z^{12})^{\frac{1}{4}}}\,12z^{11}\,\mathrm{d}z = 12\int \frac{z^6}{z^8-z^3}\,z^{11}\,\mathrm{d}z = 12\int \frac{z^{17}}{z^3(z^5-1)}\,\mathrm{d}z =$$

$$= 12\int \frac{z^{14}}{z^5-1}\,\mathrm{d}z = 12\int \frac{z^{10}}{z^5-1}\,z^4\,\mathrm{d}z = \left| \begin{array}{r} z^5-1 = t \\ z^5 = t+1 \\ 5z^4\,\mathrm{d}z = \mathrm{d}t \\ z^4\,\mathrm{d}z = \frac{1}{5}\,\mathrm{d}t \\ z^{10} = (z^5)^2 \end{array} \right| =$$

$$= 12\int \frac{(t+1)^2}{t}\cdot\frac{1}{5}\,\mathrm{d}t = \frac{12}{5}\int \frac{t^2+2t+1}{t}\,\mathrm{d}t = \frac{12}{5}\int \left(t+2+\frac{1}{t}\right)\mathrm{d}t =$$

$$= \frac{12}{5}\left(\frac{t^2}{2}+2t+\ln|t|\right)+c = \frac{6}{5}\left[(z^5-1)^2+4(z^5-1)+2\ln|(z^5-1)|\right]+c =$$

$$= \underline{\underline{\frac{6}{5}\left[\left(\sqrt[12]{x^5}-1\right)^2+4\left(\sqrt[12]{x^5}-1\right)+\ln\left(\sqrt[12]{x^5}-1\right)^2\right]+c}}$$

4. [$x \in (-1\,;\,5)$] $\underline{\displaystyle\int \frac{1}{\sqrt{4x+5-x^2}}\,\mathrm{d}x} = \displaystyle\int \frac{1}{\sqrt{5-(x^2-4x)}}\,\mathrm{d}x =$

$$= \int \frac{1}{\sqrt{5-(x^2-4x+4-4)}}\,\mathrm{d}x = \int \frac{1}{\sqrt{5-(x^2-4x+4)+4}}\,\mathrm{d}x =$$

$$= \int \frac{1}{\sqrt{9-(x-2)^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{9\left[1-\frac{(x-2)^2}{9}\right]}}\,\mathrm{d}x = \int \frac{1}{\sqrt{9}\cdot\sqrt{1-\left(\frac{x-2}{3}\right)^2}}\,\mathrm{d}x =$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{1-\left(\frac{x-2}{3}\right)^2}}\,\mathrm{d}x = \left| \begin{array}{r} \frac{x-2}{3} = y \\ x = 3y+2 \\ \mathrm{d}x = 3\,\mathrm{d}y \end{array} \right| = \frac{1}{3}\int \frac{1}{\sqrt{1-y^2}}\cdot 3\,\mathrm{d}y = \arcsin y + c =$$

$$= \underline{\underline{\arcsin\frac{x-2}{3}+c}}$$

5. [$x \in (-2\,;\,2)$]

$$\underline{\int \frac{x^2}{\sqrt{4-x^2}}\,\mathrm{d}x} = \left| \begin{array}{c} x = 2\sin y \\ \mathrm{d}x = 2\cos y\,\mathrm{d}y \\ \frac{x}{2} = \sin y \\ \arcsin\frac{x}{2} = y \end{array} \right| =$$

$$= \int \frac{(2\sin y)^2}{\sqrt{4-(2\sin y)^2}} \cdot 2\cos y\ \mathrm{d}y = \int \frac{4\sin^2 y}{\sqrt{4-4\sin^2 y}} \cdot 2\cos y\ \mathrm{d}y =$$

$$= \int \frac{8\sin^2 y \cdot \cos y}{\sqrt{4}\cdot\sqrt{1-\sin^2 y}}\ \mathrm{d}y = \int \frac{8\sin^2 y \cdot \cos y}{2\cdot\sqrt{\cos^2 y}}\ \mathrm{d}y =$$

$$= 4\int \frac{\sin^2 y \cdot \cos y}{|\cos y|}\ \mathrm{d}y = \left| \begin{array}{c} \arcsin\frac{x}{2} = y \\ H\left(\arcsin\frac{x}{2}\right) = \left\langle -\frac{\pi}{2}\,;\,\frac{\pi}{2}\right\rangle \\ \left(-\frac{\pi}{2}\,;\,\frac{\pi}{2}\right) = D(y) \\ \cos y > 0 \end{array} \right| = 4\int \frac{\sin^2 y \cdot \cos y}{\cos y}\ \mathrm{d}y =$$

$$= 4\int \sin^2 y\ \mathrm{d}y = 4\int \frac{1-\cos 2y}{2}\ \mathrm{d}y = 2\int \mathrm{d}y - \int \cos 2y \cdot \underline{2\,\mathrm{d}y} = \left| \begin{array}{c} 2y = z \\ \underline{2\,\mathrm{d}y} = \mathrm{d}z \end{array} \right| =$$

$$= 2y - \int \cos z\ \mathrm{d}z = 2\arcsin\frac{x}{2} - \sin z + c = 2\arcsin\frac{x}{2} - \sin 2y + c =$$

$$= 2\arcsin\frac{x}{2} - 2\sin y \cdot \cos y + c = 2\arcsin\frac{x}{2} - 2\cdot\frac{x}{2}\cdot\sqrt{1-\sin^2 y} + c =$$

$$= 2\arcsin\frac{x}{2} - x\cdot\sqrt{1-\left(\frac{x}{2}\right)^2} + c = 2\arcsin\frac{x}{2} - x\cdot\sqrt{\frac{4-x^2}{4}} + c =$$

$$= \underline{\underline{2\arcsin\frac{x}{2} - \frac{x}{2}\cdot\sqrt{4-x^2} + c}}$$

6. a) [$x \in (-\infty\,;\,-2\rangle \cup \langle 2;\,\infty)$] $\underline{\int \sqrt{x^2-4}\,\mathrm{d}x =}$

$$= \left| \begin{aligned} \sqrt{x^2-4} &= v - x \\ x^2 - 4 &= v^2 - 2vx + x^2 \\ 2vx &= v^2 + 4 \\ x &= \frac{v^2+4}{2v} \\ \mathrm{d}x &= \frac{2v(2v)-(v^2+4)2}{(2v)^2}\,\mathrm{d}v \\ \mathrm{d}x &= \frac{v^2-4}{2v^2}\,\mathrm{d}v \\ x + \sqrt{x^2-4} &= v \end{aligned} \right| = \int \left(v - \frac{v^2+4}{2v}\right) \cdot \frac{v^2-4}{2v^2}\,\mathrm{d}v =$$

$$= \int \frac{[2v^2-(v^2+4)]\cdot(v^2-4)}{2v\cdot 2v^2}\,\mathrm{d}v = \int \frac{(v^2-4)^2}{4v^3}\,\mathrm{d}v = \int \frac{v^4-8v^2+16}{4v^3}\,\mathrm{d}v =$$

$$= \int \frac{v^4}{4v^3}\,\mathrm{d}v - \frac{8}{4}\int \frac{v^2}{v^3}\,\mathrm{d}v + \int \frac{16}{4v^3}\,\mathrm{d}v = \frac{1}{4}\int v\,\mathrm{d}v - 2\int \frac{1}{v}\,\mathrm{d}v + 4\int v^{-3}\,\mathrm{d}v =$$

$$= \frac{1}{4}\cdot\frac{v^2}{2} - 2\ln|v| + 4\cdot\frac{v^{-2}}{-2} + c = \frac{v^2}{8} - \ln v^2 - \frac{2}{v^2} + c =$$

$$= \frac{v^4-16}{8v^2} - \ln v^2 + c = \frac{\left(x+\sqrt{x^2-4}\right)^4 - 16}{8\left(\sqrt{x^2-4}+x\right)^2} - \ln\left(x+\sqrt{x^2-4}\right)^2 + c = \ldots =$$

$$= \underline{\underline{\frac{x}{2}\cdot\sqrt{x^2-4} - \ln\left(x+\sqrt{x^2-4}\right)^2 + c}}\ {}^{1}$$

[1] Tento výsledek dává webová stránka
`https://www.wolframalpha.com/input/?i=integrate+sqrt(x^2-4)`

6. b) $[x \in (-\infty; -2\rangle \cup \langle 2; \infty)]$ $\int \sqrt{x^2-4}\,\mathrm{d}x =$

$$\left|\begin{aligned} x &= \frac{2}{\sin w} = 2\sin^{-1} w \\ \mathrm{d}x &= -2\sin^{-2} w \cdot \cos w \;\; \mathrm{d}w \\ \mathrm{d}x &= \frac{-2\cos w}{\sin^2 w} \;\; \mathrm{d}w \\ \sin w &= \frac{2}{x} \\ w &= \arcsin\frac{2}{x} \quad \Rightarrow w \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right) \end{aligned}\right|$$

$$= \int \sqrt{\left(\frac{2}{\sin w}\right)^2 - 4} \cdot \frac{-2\cos w}{\sin^2 w} \;\; \mathrm{d}w = \int \sqrt{\frac{4(1-\sin^2 w)}{\sin^2 w}} \cdot \frac{-2\cos w}{\sin^2 w} \;\; \mathrm{d}w =$$

$$= \int \frac{\sqrt{4}\cdot\sqrt{1-\sin^2 w}}{\sqrt{\sin^2 w}} \cdot \frac{-2\cos w}{\sin^2 w} \;\; \mathrm{d}w = \left|\begin{aligned} \sqrt{1-\sin^2 w} &= z \\ 1-\sin^2 w &= z^2 \\ \sin^2 w &= 1-z^2 \\ \sin w &= \sqrt{1-z^2} \\ \sqrt{\cos^2 w} &= z \\ |\cos w| &= z \\ w \in \ldots : \quad \cos w &= z \\ -\sin w\,\mathrm{d}w &= \mathrm{d}z \\ \mathrm{d}w &= \frac{-1}{\sqrt{1-z^2}} \;\; \mathrm{d}z \end{aligned}\right| =$$

$$= \int \frac{2z}{\sqrt{1-z^2}} \cdot \frac{-2z}{1-z^2} \cdot \frac{-1}{\sqrt{1-z^2}} \;\; \mathrm{d}z = \int \frac{4z^2}{(1-z^2)^2} \;\; \mathrm{d}z = \int \frac{4z^2}{(z^2-1)^2} \;\; \mathrm{d}z =$$

$$= \int \frac{4z^2}{[(z+1)(z-1)]^2} \;\; \mathrm{d}z = \int \left[\frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{z-1} + \frac{D}{(z-1)^2}\right] \;\; \mathrm{d}z =$$

$$\left|\begin{array}{lll} & 4z^2 = A(z+1)(z-1)^2 + B(z-1)^2 + C(z+1)^2(z-1) + D(z+1)^2 & \\ z=1: & 4 = 4D & D = 1 \\ z=-1: & 4 = 4B & B = 1 \\ z^3: & 0 = A + C \qquad A = -C & \\ z=0: & 0 = A + B - C + D & \\ & 0 = A + 1 + A + 1 & A = -1 \\ & & C = 1 \end{array}\right|$$

$$= \int \left[\frac{-1}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{z-1} + \frac{1}{(z-1)^2}\right] \;\; \mathrm{d}z =$$

$$= -\int \frac{(z+1)'}{z+1}\ \mathrm{d}z + \int \frac{1}{(z+1)^2}\ \mathrm{d}z + \int \frac{(z-1)'}{z-1}\ \mathrm{d}z + \int \frac{1}{(z-1)^2}\ \mathrm{d}z =$$

$$= \left| \begin{array}{r} z+1 = u \\ \mathrm{d}z = \mathrm{d}u \end{array} \right| \begin{array}{r} z-1 = v \\ \mathrm{d}z = \mathrm{d}v \end{array} \left| = -\ln|z+1| + \int \frac{1}{u^2}\ \mathrm{d}u + \ln|z-1| + \int \frac{1}{v^2}\ \mathrm{d}v = \right.$$

$$= \ln\frac{|z-1|}{|z+1|} + \int u^{-2}\,\mathrm{d}u + \int v^{-2}\,\mathrm{d}v = \ln\frac{|z-1|}{|z+1|} + \frac{u^{-1}}{-1} + \frac{v^{-1}}{-1} + c =$$

$$= \ln\left|\frac{z-1}{z+1}\right| - \frac{1}{u} - \frac{1}{v} + c = \ln\left|\frac{z-1}{z+1}\cdot\frac{z-1}{z-1}\right| - \frac{1}{z+1} - \frac{1}{z-1} + c =$$

$$= \ln\left|\frac{z^2-2z+1}{z^2-1}\right| - \frac{(z-1)+(z+1)}{z^2-1} + c = \ln\left|\frac{z^2-2z+1}{z^2-1}\right| - \frac{2z}{z^2-1} + c =$$

$$= \ln\left|\frac{1-\sin^2 w - 2\cdot\sqrt{1-\sin^2 w} + 1}{1-\sin^2 w - 1}\right| - \frac{2\cdot\sqrt{1-\sin^2 w}}{1-\sin^2 w - 1} + c =$$

$$= \ln\left|\frac{1-\left(\frac{2}{x}\right)^2 - 2\cdot\sqrt{1-\left(\frac{2}{x}\right)^2} + 1}{1-\left(\frac{2}{x}\right)^2 - 1}\right| - \frac{2\cdot\sqrt{1-\left(\frac{2}{x}\right)^2}}{1-\left(\frac{2}{x}\right)^2 - 1} + c =$$

$$= \ln\left|\left(1-\frac{4}{x^2} - 2\cdot\sqrt{1-\frac{4}{x^2}} + 1\right)\cdot\frac{x^2}{-4}\right| - \left(2\cdot\sqrt{1-\frac{4}{x^2}}\right)\cdot\frac{x^2}{-4} + c =$$

$$= \underline{\underline{\frac{x}{2}\cdot\sqrt{x^2-4} - \ln\left(x+\sqrt{x^2-4}\right)^2 + c}}$$

6. c) $[x \in (-\infty\,;\,-2\rangle \cup \langle 2;\,\infty)]$ $\quad \underline{\int \sqrt{x^2-4}\ \mathrm{d}x} = \left| \begin{array}{c} x = \mathrm{e}^y + \mathrm{e}^{-y} \\ \mathrm{d}x = (\mathrm{e}^y - \mathrm{e}^{-y})\,\mathrm{d}y \end{array} \right| =$

$$= \int \sqrt{(\mathrm{e}^y + \mathrm{e}^{-y})^2 - 4}\cdot(\mathrm{e}^y - \mathrm{e}^{-y})\ \ \mathrm{d}y =$$

$$= \int \sqrt{\mathrm{e}^{2y} + 2\,\underbrace{\mathrm{e}^{2y}\mathrm{e}^{-2y}}_{\mathrm{e}^0=1} + \mathrm{e}^{-2y} - 4}\cdot(\mathrm{e}^y - \mathrm{e}^{-y})\ \ \mathrm{d}y =$$

$$= \int \sqrt{(\mathrm{e}^y - \mathrm{e}^{-y})^2} \cdot (\mathrm{e}^y - \mathrm{e}^{-y})\,\mathrm{d}y = \underbrace{\int |\mathrm{e}^y - \mathrm{e}^{-y}| \cdot (\mathrm{e}^y - \mathrm{e}^{-y})\,\mathrm{d}y}_{\mathcal{J}} =$$

$$\mathcal{A}: y > 0 \quad \Longrightarrow \quad \mathcal{J} = \int (\mathrm{e}^y - \mathrm{e}^{-y}) \cdot (\mathrm{e}^y - \mathrm{e}^{-y})\,\mathrm{d}y = \int (\mathrm{e}^y - \mathrm{e}^{-y})^2\,\mathrm{d}y =$$

$$= \int (\mathrm{e}^{2y} - 2 + \mathrm{e}^{-2y})\,\mathrm{d}y = \frac{1}{2} \cdot \mathrm{e}^{2y} - 2y + \frac{1}{-2} \cdot \mathrm{e}^{-2y} + c_1 =$$

Jestliže: $x = \mathrm{e}^y + \mathrm{e}^{-y}$ pak: $y = \ln\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right)$, protože:

$$x = \mathrm{e}^{\ln \frac{x+\sqrt{x^2-4}}{2}} + \mathrm{e}^{-\ln \frac{x+\sqrt{x^2-4}}{2}} = \mathrm{e}^{\ln \frac{x+\sqrt{x^2-4}}{2}} + \mathrm{e}^{\ln\left(\frac{x+\sqrt{x^2-4}}{2}\right)^{-1}} =$$

$$= \frac{x + \sqrt{x^2 - 4}}{2} + \frac{2}{x + \sqrt{x^2 - 4}} = \frac{x + \sqrt{x^2 - 4}}{2} + \frac{2}{x + \sqrt{x^2 - 4}} \cdot \frac{x - \sqrt{x^2 - 4}}{x - \sqrt{x^2 - 4}} =$$

$$= \frac{x + \sqrt{x^2 - 4}}{2} + \frac{2 \cdot \left(x - \sqrt{x^2 - 4}\right)}{x^2 - (x^2 - 4)} = \frac{x + \sqrt{x^2 - 4}}{2} + \frac{2 \cdot \left(x - \sqrt{x^2 - 4}\right)}{4} = x$$

$$= \frac{1}{2} \cdot \mathrm{e}^{2\ln\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right)} - 2\ln\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right) - \frac{1}{2} \cdot \mathrm{e}^{-2\ln\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right)} + c_1 =$$

$$= \frac{1}{2} \cdot \left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right)^2 - \ln\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right)^2 - \frac{1}{2 \cdot \left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2 - 4}\right)^2} + c_1 =$$

$$= \frac{1}{8} \cdot \left(x + \sqrt{x^2 - 4}\right)^2 - \frac{2}{\left(x + \sqrt{x^2 - 4}\right)^2} \cdot \frac{\left(x - \sqrt{x^2 - 4}\right)^2}{\left(x - \sqrt{x^2 - 4}\right)^2} -$$

$$- \ln\left[\left(x + \sqrt{x^2 - 4}\right) \cdot \frac{1}{2}\right]^2 + c_1 =$$

$$= \frac{1}{8} \cdot \left(x + \sqrt{x^2 - 4}\right)^2 - \frac{2 \cdot \left(x^2 - \sqrt{x^2 - 4}\right)^2}{[x^2 - (x^2 - 4)]^2} - \ln\left(x + \sqrt{x^2 - 4}\right)^2 + \underbrace{\ln 4 + c_1}_{c} =$$

$$= \frac{1}{8} \cdot \left(x + \sqrt{x^2 - 4}\right)^2 - \frac{\left(x^2 - \sqrt{x^2 - 4}\right)^2}{8} - \ln\left(x + \sqrt{x^2 - 4}\right)^2 + c =$$

$$= \frac{x^2 + 2x\sqrt{x^2 - 4} + (x^2 - 4) - \left[x^2 - 2x\sqrt{x^2 - 4} + (x^2 - 4)\right]}{8} - \ln\left(x + \sqrt{x^2 - 4}\right)^2 + c =$$

$$= \underline{\underline{\frac{x}{2} \cdot \sqrt{x^2 - 4} - \ln\left(x + \sqrt{x^2 - 4}\right)^2 + c}} =$$

$$\mathcal{B} : y < 0 \quad \Longrightarrow \quad \mathcal{J} = \int -(\mathrm{e}^{y} - \mathrm{e}^{-y}) \cdot (\mathrm{e}^{y} - \mathrm{e}^{-y})\,\mathrm{d}y =$$

$$= -\int (\mathrm{e}^{2y} - 2 + \mathrm{e}^{-2y})\,\mathrm{d}y = -\int (\mathrm{e}^{y} - \mathrm{e}^{-y})^2\,\mathrm{d}y = -\int (\mathrm{e}^{2y} - 2 + \mathrm{e}^{-2y})\,\mathrm{d}y =$$

$$= -\frac{1}{2} \cdot \mathrm{e}^{2y} + 2y - \frac{1}{-2} \cdot \mathrm{e}^{-2y} + c_2 =$$

$$= -\frac{1}{2} \cdot \mathrm{e}^{2\ln\left(\frac{x}{2} + \frac{1}{2}\cdot\sqrt{x^2-4}\right)} + 2\ln\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2-4}\right) + \frac{1}{2} \cdot \mathrm{e}^{-2\ln\left(\frac{x}{2} + \frac{1}{2}\cdot\sqrt{x^2-4}\right)} + c_2 =$$

$$= -\frac{1}{2}\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2-4}\right)^2 + \ln\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2-4}\right)^2 + \frac{1}{2}\left(\frac{x}{2} + \frac{1}{2} \cdot \sqrt{x^2-4}\right)^{-2} + c_2 =$$

$$= \qquad \ldots \qquad = \underline{\underline{\frac{x}{2} \cdot \sqrt{x^2-4} - \ln\left(x + \sqrt{x^2-4}\right)^2 + c}}$$

6. d) [$x \in (-\infty\,;\,-2\rangle \cup \langle 2;\,\infty)$] $\quad \underline{\int \sqrt{x^2-4}\,\mathrm{d}x} = \int \sqrt{(x+2)\cdot(x-2)}\,\mathrm{d}x =$

$$= \underbrace{\int \sqrt{(x+2)^2 \cdot \frac{x-2}{x+2}}}_{\text{pro } x \neq -2}\,\mathrm{d}x =$$

$$\left|\begin{array}{l}
\sqrt{\frac{x-2}{x+2}} = u\,, \quad \text{kde } u \in \langle 0\,;1)\,, \text{pro } x \in \langle 2\,;\infty) \\
\qquad\qquad\quad \text{a } u \in (1\,;\infty)\,, \text{pro } x \in (-\infty\,;-2) \\
\frac{x-2}{x+2} = u^2 \\
x - 2 = u^2(x+2) \\
x - u^2 x = 2u^2 + 2 \\
x(1-u^2) = 2u^2 + 2 \\
x = \frac{2u^2+2}{1-u^2} \\
\mathrm{d}x = \frac{4u(1-u^2)-(2u^2+2)(-2u)}{(1-u^2)^2}\,\mathrm{d}u \\
\mathrm{d}x = \frac{8u}{(1-u^2)^2}\,\mathrm{d}u
\end{array}\right|$$

$$= \int \sqrt{\left(\frac{2u^2+2}{1-u^2} + 2\right)^2 \cdot u^2} \cdot \frac{8u}{(1-u^2)^2}\,\mathrm{d}u = \int \sqrt{\left(\frac{4}{1-u^2}\right)^2} \cdot \sqrt{u^2} \cdot \frac{8u}{(1-u^2)^2}\,\mathrm{d}u =$$

$$= \int \frac{4}{|1-u^2|} \cdot u \cdot \frac{8u}{(1-u^2)^2}\,\mathrm{d}u = \underbrace{\int \frac{32u^2}{|1-u^2| \cdot (1-u^2)^2}\,\mathrm{d}u}_{\mathcal{J}} =$$

$\mathcal{A} : u \in \langle 0\,;1) \quad \Longrightarrow \quad \mathcal{I} = \int \frac{32u^2}{(1-u^2)\cdot(1-u^2)^2}\,\mathrm{d}u = \int \frac{32u^2}{(1-u^2)^3}\,\mathrm{d}u =$

$$= \int \frac{32u^2}{[(1+u)(1-u)]^3}\,\mathrm{d}u$$

$$\frac{32u^2}{[(1+u)(1-u)]^3} = \frac{A}{1+u} + \frac{B}{(1+u)^2} + \frac{C}{(1+u)^3} + \frac{D}{1-u} + \frac{E}{(1-u)^2} + \frac{F}{(1-u)^3}$$

$$32u^2 = A(1+u)^2(1-u)^3 + B(1+u)(1-u)^3 + C(1-u)^3 + D(1+u)^3(1-u)^2 + E(1+u)^3(1-u) + F(1+u)^3$$

$u = -1 : 32 = 8C \quad \Longrightarrow \quad C = 4$

$u = 1 : 32 = 8F \quad \Longrightarrow \quad F = 4$

$32u^2 = A(1+2u+u^2)(1-u)^3 + B(1+u)(1-u)^3 + C(1-3u+3u^2-u^3) +$

$+ D(1+u)^3(1-2u+u^2) + E(1+u)^3(1-u) + F(1+3u+3u^2+u^3)$

$u^5 : \quad 0 = -5A + 5D \qquad A = D$

$u^2 : 32 = A(3-6+1) + B(3-3) + C(3) + D(3-6+1) + E(-3+3) + F(3)$

$32 = -2D + 4\cdot 3 - 2D + 4\cdot 3 \qquad A, D = -2$

$u^4 : \quad 0 = A(3-2) + B(-1) + D(3-2) + E(-1)$

$0 = -2\cdot 1 - B - 2\cdot 1 - E \qquad B = -4 - E$

$u^1 : \quad 0 = A(2-3) + B(1-3) + C(-3) + D(3-2) + E(3-1) + F(3)$

$0 = -2\cdot(-1) + (-4-E)\cdot(-2) + 4\cdot(-3) - 2\cdot 1 + 2E + 4\cdot 3 \qquad B, E = -2$

$$= \int \left[\frac{-2}{1+u} + \frac{-2}{(1+u)^2} + \frac{4}{(1+u)^3} + \frac{-2}{1-u} + \frac{-2}{(1-u)^2} + \frac{4}{(1-u)^3}\right]\mathrm{d}u =$$

$$= \left| \begin{array}{r} 1+u = v \\ \mathrm{d}u = dv \end{array} \right| \begin{array}{r} 1-u = w \\ -\mathrm{d}u = \mathrm{d}w \end{array} \right| =$$

$$= -2\int \frac{1}{v}\,\mathrm{d}v - 2\int v^{-2}\,\mathrm{d}v + 4\int v^{-3}\,\mathrm{d}v + 2\int \frac{1}{w}\,\mathrm{d}w + 2\int w^{-2}\,\mathrm{d}w - 4\int w^{-3}\,\mathrm{d}w =$$

$$= -2\ln|v| - 2\frac{v^{-1}}{-1} + 4\frac{v^{-2}}{-2} + 2\ln|w| + 2\frac{w^{-1}}{-1} - 4\frac{w^{-2}}{-2} + c =$$

$$= \frac{2}{v} - \frac{2}{v^2} - \frac{2}{w} + \frac{2}{w^2} - \ln v^2 + \ln w^2 + c = \frac{2v-2}{v^2} + \frac{-2w+2}{w^2} - (\ln v^2 - \ln w^2) + c =$$

$$= \frac{2v-2}{v^2} - \frac{2w-2}{w^2} - \ln\frac{v^2}{w^2} + c = \frac{2(1+u)-2}{(1+u)^2} - \frac{2(1-u)-2}{(1-u)^2} - \ln\frac{(1+u)^2}{(1-u)^2} + c =$$

$$= \frac{2u}{(1+u)^2} + \frac{2u}{(1-u)^2} - \ln\left(\frac{1+u}{1-u}\right)^2 + c =$$

$$= \frac{2\cdot\sqrt{\frac{x-2}{x+2}}}{\left(1+\sqrt{\frac{x-2}{x+2}}\right)^2} + \frac{2\cdot\sqrt{\frac{x-2}{x+2}}}{\left(1-\sqrt{\frac{x-2}{x+2}}\right)^2} - \ln\left(\frac{1+\sqrt{\frac{x-2}{x+2}}}{1-\sqrt{\frac{x-2}{x+2}}}\right)^2 + c = \ldots =$$

$$= \underline{\underline{\frac{x}{2}\cdot\sqrt{x^2-4} - \ln\left(2x + 2\cdot\sqrt{x^2-4}\right)^2 + c}}$$ [2]

$\mathcal{B} : u \in (1\,;\infty) \quad \Longrightarrow \quad \mathcal{I} = \int \ldots$

[2] Tento výsledek dává webová stránka
`http://um.mendelu.cz/maw-html/index.php?lang=cs&form=integral&function=sqrt(x^2-4)`

7. $[x \in \langle -2; -1) \cup (-1; \infty)]$

$$\underline{\int \frac{x+1}{x+\sqrt{x+2}}\,\mathrm{d}x} = \left| \begin{array}{r} \sqrt{x+2} = v \\ x+2 = v^2 \\ x = v^2-2 \\ \mathrm{d}x = 2v\,\mathrm{d}v \end{array} \right| =$$

$$= \int \frac{(v^2-2)+1}{(v^2-2)+v} \cdot 2v\,\mathrm{d}v = \int \frac{v^2-1}{v^2+v-2} \cdot 2v\,\mathrm{d}v = \int \frac{(v+1)(v-1)\cdot 2v}{(v+2)(v-1)}\,\mathrm{d}v =$$

$$= \int \frac{(v+1)\cdot 2v}{v+2}\,\mathrm{d}v = \int \frac{[(v+2)-1]\cdot 2v}{v+2}\,\mathrm{d}v =$$

$$= \int \frac{(v+2)\cdot 2v}{v+2}\,\mathrm{d}v + \int \frac{[-1]\cdot 2v}{v+2}\,\mathrm{d}v = 2\int v\,\mathrm{d}v + \int \frac{-2[(v+2)-2]}{v+2}\,\mathrm{d}v =$$

$$= 2\cdot\frac{v^2}{2} + \int \frac{-2(v+2)}{v+2}\,\mathrm{d}v + \int \frac{-2[-2]}{v+2}\,\mathrm{d}v = v^2 - 2\int \mathrm{d}v + 4\int \frac{(v+2)'}{v+2}\,\mathrm{d}v =$$

$$= v^2 - 2v + 4\ln|v+2| + c = \underline{\underline{x+2-2\cdot\sqrt{x+2} + \ln\left(2+\sqrt{x+2}\right)^4 + c}}$$

8. $[x \in \langle 0; \infty)]$

$$\underline{\int \frac{\sqrt{x}}{1+\sqrt[4]{x^3}}\,\mathrm{d}x} = \left| \begin{array}{l} \sqrt[4]{x} = y \quad \text{(viz př. 3)} \\ x = y^4 \\ \mathrm{d}x = 4y^3\,\mathrm{d}y \\ x^{\frac{1}{6}} = y \end{array} \right| = \int \frac{\sqrt{y^4}}{1+\sqrt[4]{(y^4)^3}} \cdot 4y^3\,\mathrm{d}y =$$

$$= \int \frac{y^2}{1+y^3} \cdot 4y^3\,\mathrm{d}y = \int \frac{4y^5}{y^3+1}\,\mathrm{d}y = \int \frac{4y^2[(y^3+1)-1]}{y^3+1}\,\mathrm{d}y =$$

$$= \int \frac{4y^2(y^3+1)}{y^3+1}\,\mathrm{d}y + \int \frac{4y^2[-1]}{y^3+1}\,\mathrm{d}y = 4\int y^2\,\mathrm{d}y - \frac{4}{3}\int \frac{3y^2}{y^3+1}\,\mathrm{d}y =$$

$$= \left| \begin{array}{r} y^3 = z \\ 3y^2\,\mathrm{d}y = \mathrm{d}z \\ \mathrm{d}y = \dfrac{\mathrm{d}z}{3y^2} \end{array} \right| = 4\frac{y^3}{3} - \frac{4}{3}\int \frac{3y^2}{z+1}\cdot\frac{1}{3y^2}\,\mathrm{d}z = 4\frac{y^3}{3} - \frac{4}{3}\int \frac{(z+1)'}{z+1}\,\mathrm{d}z =$$

$$= 4\frac{y^3}{3} - \frac{4}{3}\ln|z+1| + c = \frac{4}{3}y^3 - \frac{1}{3}\ln\left(y^3+1\right)^4 + c =$$

$$= \underline{\underline{\frac{4}{3}\cdot\sqrt[4]{x^3} - \frac{1}{3}\ln\left(\sqrt[4]{x^3}+1\right)^4 + c}}$$

9. $\left[x \in \mathbb{R} \setminus \left\{-\frac{5}{3}\right\}\right]$

$$\int \frac{\sqrt[3]{3x+4}}{1+\sqrt[3]{3x+4}}\,\mathrm{d}x = \left| \begin{array}{r} \sqrt[3]{3x+4} = u \\ 3x+4 = u^3 \\ 3\,\mathrm{d}x = 3u^2\,\mathrm{d}u \\ \mathrm{d}x = u^2\,\mathrm{d}u \end{array} \right| = \int \frac{u}{1+u}\cdot u^2\,\mathrm{d}u =$$

$$= \int \frac{u^2[(u+1)-1]}{u+1}\,\mathrm{d}u = \int \frac{u^2(u+1)}{u+1}\,\mathrm{d}u + \int \frac{u^2[-1]}{u+1}\,\mathrm{d}u =$$

$$= \int u^2\,\mathrm{d}u + \int \frac{(-u)u}{u+1}\,\mathrm{d}u = \int u^2\,\mathrm{d}u + \int \frac{(-u)[(u+1)-1]}{u+1}\,\mathrm{d}u =$$

$$= \frac{u^3}{3} + \int \frac{(-u)(u+1)}{u+1}\,\mathrm{d}u + \int \frac{(-u)[-1]}{u+1}\,\mathrm{d}u = \frac{1}{3}u^3 - \int u\,\mathrm{d}u + \int \frac{u}{u+1}\,\mathrm{d}u =$$

$$= \frac{1}{3}u^3 - \frac{u^2}{2} + \int \frac{(u+1)-1}{u+1}\,\mathrm{d}u = \frac{1}{3}u^3 - \frac{1}{2}u^2 + \int \frac{(u+1)}{u+1}\,\mathrm{d}u + \int \frac{-1}{u+1}\,\mathrm{d}u =$$

$$= \frac{1}{3}u^3 - \frac{1}{2}u^2 + \int \mathrm{d}u - \int \frac{(u+1)'}{u+1}\,\mathrm{d}u = \frac{1}{3}u^3 - \frac{1}{2}u^2 + u - \ln|u+1| + c =$$

$$= \frac{1}{3}(3x+4) - \frac{1}{2}\cdot\sqrt[3]{(3x+4)^2} + \sqrt[3]{3x+4} - \ln\left|1+\sqrt[3]{3x+4}\right| + c$$

10. $\left[x \in \mathbb{R} \setminus \{1 \pm \sqrt{6}\}\right]$

$$\int \frac{8x-11}{\sqrt{5+2x-x^2}}\,\mathrm{d}x = \int \frac{-4\left(-2x+\frac{11}{4}\right)}{\sqrt{5+2x-x^2}}\,\mathrm{d}x =$$

$$= \int \frac{-4\left[(2-2x)-2+\frac{11}{4}\right]}{\sqrt{5+2x-x^2}}\,\mathrm{d}x = \int \frac{-4(2-2x)}{\sqrt{5+2x-x^2}}\,\mathrm{d}x + \int \frac{-4\left[\frac{3}{4}\right]}{\sqrt{5+2x-x^2}}\,\mathrm{d}x =$$

$$= \left| \begin{array}{r} \sqrt{5+2x-x^2} = v \\ 5+2x-x^2 = v^2 \\ (2-2x)\,\mathrm{d}x = 2v\,\mathrm{d}v \\ \mathrm{d}x = \frac{2v}{2-2x}\,\mathrm{d}v \end{array} \right| = \int \frac{-4(2-2x)}{v}\cdot\frac{2v}{2-2x}\,\mathrm{d}v + \int \frac{-3}{\sqrt{5+2x-x^2}}\,\mathrm{d}x =$$

$$= \int -8\,\mathrm{d}v + \int \frac{-3}{\sqrt{5-(x^2-2x)}}\,\mathrm{d}x = -8\int \mathrm{d}v - 3\int \frac{1}{\sqrt{5-(x^2-2x+1-1)}}\,\mathrm{d}x =$$

$$= -8v - 3\int \frac{1}{\sqrt{5-(x^2-2x+1)+1}}\,\mathrm{d}x = -8v - 3\int \frac{1}{\sqrt{6-(x-1)^2}}\,\mathrm{d}x =$$

$$= -8v - 3\int \frac{1}{\sqrt{6\left[1-\frac{(x-1)^2}{6}\right]}}\,\mathrm{d}x = -8v - 3\int \frac{1}{\sqrt{6}\cdot\sqrt{1-\left(\frac{x-1}{\sqrt{6}}\right)^2}}\,\mathrm{d}x =$$

$$= \left| \begin{array}{r} \dfrac{x-1}{\sqrt{6}} = w \\ x-1 = w\sqrt{6} \\ \mathrm{d}x = \sqrt{6}\,\mathrm{d}w \end{array} \right| = -8v - 3\int \frac{\sqrt{6}}{\sqrt{6}\cdot\sqrt{1-w^2}}\,\mathrm{d}w = -8v - 3\arcsin w + c =$$

$$= \underline{\underline{-8\sqrt{5+2x-x^2} - 3\arcsin\left(\frac{x-1}{\sqrt{6}}\right) + c}}$$

11. a) [$x \in (-\infty; -1\rangle \cup (0; \infty)$] $\underline{\int \frac{1}{x+\sqrt{x^2+x}}\,\mathrm{d}x} = \int \frac{1}{x+\sqrt{x(x+1)}}\,\mathrm{d}x =$

$$= \int \frac{1}{x+\sqrt{x^2\cdot\frac{x+1}{x}}}\,\mathrm{d}x = \int \frac{1}{x+\sqrt{x^2}\cdot\sqrt{\frac{x+1}{x}}}\,\mathrm{d}x = \left| \begin{array}{r} \sqrt{\frac{x+1}{x}} = z \\ \frac{x+1}{x} = z^2 \\ x+1 = x\cdot z^2 \\ 1 = x\cdot z^2 - 1 \\ x = \dfrac{1}{z^2-1} \\ \mathrm{d}x = \dfrac{-2z}{\left(z^2-1\right)^2}\,\mathrm{d}z \end{array} \right| =$$

$$= \int \frac{1}{\frac{1}{z^2-1}+\sqrt{\left(\frac{1}{z^2-1}\right)^2}\cdot z}\cdot\frac{-2z}{\left(z^2-1\right)^2}\,\mathrm{d}z =$$

$$= \int \frac{1}{\left[\frac{1}{z^2-1}+z\cdot\sqrt{\left(\frac{1}{z^2-1}\right)^2}\right]\cdot(z^2-1)}\cdot\frac{-2z}{z^2-1}\,\mathrm{d}z =$$

$$= \int \frac{1}{1+z\cdot\sqrt{\frac{1}{\left(z^2-1\right)^2}\cdot\left(z^2-1\right)^2}}\cdot\frac{-2z}{z^2-1}\,\mathrm{d}z = \int \frac{1}{1+z\sqrt{1}}\cdot\frac{-2z}{(z+1)(z-1)}\,\mathrm{d}z =$$

$$= \int \frac{-2z}{(z+1)^2(z-1)}\,\mathrm{d}z = \int \left[\frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{z-1}\right] \mathrm{d}z =$$

$$= \left| \begin{array}{lll} & -2z = A(z+1)(z-1) + B(z-1) + C(z+1)^2 & \\ z = 1: & -2 = 4C & C = \frac{-1}{2} \\ z = -1: & 2 = -2B & B = -1 \\ z = 0: & 0 = -A - B + C & A = \frac{1}{2} \end{array} \right| =$$

$$= \int \frac{\frac{1}{2}}{z+1}\,\mathrm{d}z + \underbrace{\int \frac{-1}{(z+1)^2}\,\mathrm{d}z}_{\mathcal{I}} + \int \frac{\frac{-1}{2}}{z-1}\,\mathrm{d}z =$$

$$= \frac{1}{2}\int \frac{(z+1)'}{z+1}\,\mathrm{d}z + \mathcal{I} - \frac{1}{2}\int \frac{(z-1)'}{z-1}\,\mathrm{d}z =$$

Výpočet integrálu $\mathcal{I}$ (nebo substituce $|\, z+1 = t \,|$)

$$\int \frac{-1}{z+1}\,\mathrm{d}z = \left| \begin{array}{ll} u = \frac{-1}{z+1} & v' = 1 \\ u' = \frac{1}{(z+1)^2} & v = z \end{array} \right| = \frac{-1}{z+1}\cdot z - \int \frac{1}{(z+1)^2}\cdot z\,\mathrm{d}z = \frac{-z}{z+1} - \int \frac{(z+1)-1}{(z+1)^2}\,\mathrm{d}z$$

$$\int \frac{-1}{z+1}\,\mathrm{d}z = \frac{-z}{z+1} - \int \frac{1}{z+1}\,\mathrm{d}z - \mathcal{I}$$

$$\mathcal{I} = \frac{-z}{z+1} + c_1$$

$$= \frac{1}{2}\ln|z+1| + \frac{-z}{z+1} - \frac{1}{2}\ln|z-1| + c_a = \frac{-z}{z+1} + \frac{1}{2}\ln\left|\frac{z+1}{z-1}\right| + c_a =$$

$$= \frac{-z}{z+1}\cdot\frac{z-1}{z-1} + \frac{1}{2}\ln\left|\frac{z+1}{z-1}\cdot\frac{z+1}{z+1}\right| + c_a = \frac{-z^2+z}{z^2-1} + \frac{1}{2}\ln\left|\frac{(z+1)^2}{z^2-1}\right| + c_a =$$

$$= \frac{-z^2+z}{z^2-1} + \frac{1}{2}\ln\left|\frac{z^2+2z+1}{z^2-1}\right| + c_a =$$

$$= \frac{-\left(\sqrt{\frac{x+1}{x}}\right)^2 + \sqrt{\frac{x+1}{x}}}{\left(\sqrt{\frac{x+1}{x}}\right)^2 - 1} + \frac{1}{2}\ln\left|\frac{\left(\sqrt{\frac{x+1}{x}}\right)^2 + 2\cdot\sqrt{\frac{x+1}{x}} + 1}{\left(\sqrt{\frac{x+1}{x}}\right)^2 - 1}\right| + c_a =$$

$$= \frac{-\frac{x+1}{x} + \sqrt{\frac{x+1}{x}}}{\frac{x+1}{x} - 1} + \frac{1}{2}\ln\left|\frac{\frac{x+1}{x} + 2\cdot\sqrt{\frac{x+1}{x}} + 1}{\frac{x+1}{x} - 1}\right| + c_a =$$

$$= \frac{-\frac{x+1}{x} + \frac{1}{x}\cdot\sqrt{x(x+1)}}{\frac{x+1-x}{x}} + \frac{1}{2}\ln\left|\frac{\frac{x+1+x}{x} + \frac{2}{x}\cdot\sqrt{x(x+1)}}{\frac{x+1-x}{x}}\right| + c_a =$$

$$= \left[\frac{-x-1+\sqrt{x^2+x}}{x}\right]\cdot\frac{x}{1} + \frac{2}{4}\ln\left|\left[\frac{2x+1+2\cdot\sqrt{x^2+x}}{x}\right]\cdot\frac{x}{1}\right| + c_a =$$

$$= -x \underbrace{-1+c_a}_{c} + \sqrt{x^2+x} + \frac{1}{4}\ln\left(2x+1+2\sqrt{x^2+x}\right)^2 =$$

$$= -x + \sqrt{x^2+x} + \frac{1}{4}\ln\left(2x+1+2\sqrt{x^2+x}\right)^2 + c$$

11. b) $[x \in (-\infty;\, -1\rangle \cup (0;\, \infty)]$ $\int \frac{1}{x+\sqrt{x^2+x}}\,\mathrm{d}x =$
$$\left|\begin{array}{l} \sqrt{x^2+x} = t - x \\ x^2+x = t^2 - 2tx + x^2 \\ 2tx + x = t^2 \\ x = \frac{t^2}{2t+1} \\ \mathrm{d}x = \frac{2t(2t+1)-2t^2}{(1+2t)^2}\,\mathrm{d}t \\ \mathrm{d}x = \frac{2t^2+2t}{(2t+1)^2}\,\mathrm{d}t \end{array}\right| =$$

$$= \int \frac{1}{x+(t-x)}\cdot\frac{2t^2+2t}{(2t+1)^2}\,\mathrm{d}t = \int \frac{t(2t+2)}{t(2t+1)^2}\,\mathrm{d}t = \int\left[\frac{A}{2t+1} + \frac{B}{(2t+1)^2}\right]\mathrm{d}t =$$

$$\left|\begin{array}{lll} & 2t+2 = A(2t+1)+B & \\ t = \frac{-1}{2}: & 1 = B & B = 1 \\ t^1: & 2 = 2A & A = 1 \end{array}\right|$$

$$= \int \frac{1}{2t+1}\,\mathrm{d}t + \underbrace{\int \frac{1}{(2t+1)^2}\,\mathrm{d}t}_{\mathcal{J}} = \frac{1}{2}\int \frac{2}{2t+1}\,\mathrm{d}t + \mathcal{J} = \frac{1}{2}\int \frac{(2t+1)'}{2t+1}\,\mathrm{d}t + \mathcal{J} =$$

Výpočet integrálu $\mathcal{J}$ (nebo substituce $|\ 2t+1=w\ |$)

$$\int \frac{1}{2t+1}\,\mathrm{d}t = \left| \begin{array}{ll} u=\frac{1}{2t+1} & v'=1 \\ u'=\frac{-2}{(2t+1)^2} & v=t \end{array} \right| = \frac{1}{2t+1}\cdot t - \int \frac{-2}{(2t+1)^2}\cdot t\,\mathrm{d}t = \frac{t}{2t+1} + \int \frac{(2t+1)-1}{(2t+1)^2}\,\mathrm{d}t$$

$$\int \frac{1}{2t+1}\,\mathrm{d}t = \frac{t}{2t+1} + \int \frac{1}{2t+1}\,\mathrm{d}t - \mathcal{J}$$

$$\mathcal{J} = \frac{t}{2t+1} + c_2$$

$$= \frac{1}{2}\ln|2t+1| + \frac{t}{2t+1} + c = \frac{t}{2t+1} + \frac{1}{4}\ln(2t+1)^2 + c =$$

$$= \frac{x+\sqrt{x^2+x}}{2\left(x+\sqrt{x^2+x}\right)+1} + \frac{1}{4}\ln\left[2\left(x+\sqrt{x^2+x}\right)+1\right]^2 + c =$$

$$= \frac{x+\sqrt{x^2+x}}{(2x+1)+2.\sqrt{x^2+x}}\cdot\frac{(2x+1)-2.\sqrt{x^2+x}}{(2x+1)-2.\sqrt{x^2+x}} + \frac{1}{4}\ln\left(2x+1+2.\sqrt{x^2+x}\right)^2 + c =$$

$$= \frac{\left(x+\sqrt{x^2+x}\right)\cdot\left(2x+1-2.\sqrt{x^2+x}\right)}{(2x+1)^2-4(x^2+x)} + \frac{1}{4}\ln\left(2x+1+2.\sqrt{x^2+x}\right)^2 + c =$$

$$= \frac{2x^2+x-2x\sqrt{x^2+x}+2x\sqrt{x^2+x}+\sqrt{x^2+x}-2(x^2+x)}{4x^2+4x+1-4x^2-4x} +$$

$$+ \frac{1}{4}\ln\left(2x+1+2\sqrt{x^2+x}\right)^2 + c =$$

$$= \frac{2x^2+x+\sqrt{x^2+x}-2x^2-2x)}{1} + \frac{1}{4}\ln\left(2x+1+2\sqrt{x^2+x}\right)^2 + c =$$

$$= \underline{\underline{-x+\sqrt{x^2+x}+\frac{1}{4}\ln\left(2x+1+2\sqrt{x^2+x}\right)^2 + c}}$$

12. [$x \in (-\infty\,; -2) \cup (1; \infty)$] $\int \frac{1}{\sqrt{x^2+x-2}}\,\mathrm{d}x =$

$$= \left| \begin{aligned} \sqrt{x^2+x-2} &= w - x \\ x^2+x-2 &= w^2-2wx+x^2 \\ 2wx+x &= w^2+2 \\ x &= \frac{w^2+2}{2w+1} \\ \mathrm{d}x &= \frac{2w(2w+1)-(w^2+2)2}{(2w+1)^2}\,\mathrm{d}w \\ \mathrm{d}x &= \frac{2w^2+2w-4}{(2w+1)^2}\,\mathrm{d}w \end{aligned} \right| = \int \frac{1}{w-\frac{w^2+2}{2w+1}} \cdot \frac{2w^2+2w-4}{(2w+1)^2}\,\mathrm{d}w =$$

$$= \int \frac{1}{\frac{2w^2+w-(w^2+2)}{2w+1}} \cdot \frac{2w^2+2w-4}{(2w+1)^2}\,\mathrm{d}w =$$

$$= \int \frac{2w+1}{w^2+w-2} \cdot \frac{2(w^2+w-2)}{(2w+1)^2}\,\mathrm{d}w = \int \frac{2}{2w+1}\,\mathrm{d}w = \int \frac{(2w+1)'}{2w+1}\,\mathrm{d}w =$$

$$= \ln|2w+1| + c = \ln\left|2\left(x+\sqrt{x^2+x-2}\right)+1\right| + c =$$

$$= \frac{1}{2}\ln\left(2x+1+2\sqrt{x^2+x-2}\right)^2 + c$$

13. [$x \in \mathbb{R}$] $\int \frac{x}{\sqrt{x^2+4x+5}}\,\mathrm{d}x =$

$$\left| \begin{aligned} \sqrt{(x+2)^2+1} &= z - x \\ x^2+4x+5 &= z^2-2xz+x^2 \\ 2xz+4x &= z^2-5 \\ x(2z+4) &= z^2-5 \\ x &= \frac{z^2-5}{2(z+2)} = \frac{z^2-5}{2z+4} \\ \mathrm{d}x &= \frac{2z[2(z+2)]-(z^2-5)\cdot 2}{[2(z+2)]^2}\,\mathrm{d}z \\ \mathrm{d}x &= \frac{2(z^2+4z+5)}{4(z+2)^2}\,\mathrm{d}z \\ \mathrm{d}x &= \frac{z^2+4z+5}{2(z+2)^2}\,\mathrm{d}z \end{aligned} \right| =$$

$$= \int \frac{\frac{z^2-5}{2z+4}}{z-\frac{z^2-5}{2z+4}} \cdot \frac{z^2+4z+5}{2(z+2)^2}\,\mathrm{d}z = \int \frac{z^2-5}{2z+4} \cdot \frac{2z+4}{2z^2+4z-z^2+5} \cdot \frac{z^2+4z+5}{2(z+2)^2}\,\mathrm{d}z =$$

$$= \int \frac{\frac{1}{2}(z^2-5)}{(z+2)^2}\,\mathrm{d}z = \int \frac{\frac{1}{2}\cdot[(z^2+4z+4)-4z-9]}{z^2+4z+4}\,\mathrm{d}z =$$

$$= \frac{1}{2}\int \frac{(z^2+4z+4)}{z^2+4z+4}\,\mathrm{d}z + \int \frac{\frac{1}{2}\cdot[-4z-9]}{(z+2)^2}\,\mathrm{d}z = \frac{1}{2}\int \mathrm{d}z + \int \frac{-2z-\frac{9}{2}}{(z+2)^2}\,\mathrm{d}z =$$

$$= \frac{1}{2}\cdot z + \int \frac{-\left[2(z+2)+\frac{1}{2}\right]}{(z+2)^2}\,\mathrm{d}z = \frac{1}{2}\cdot z + \int \frac{-2(z+2)}{(z+2)^2}\,\mathrm{d}z + \int \frac{-\left[\frac{1}{2}\right]}{(z+2)^2}\,\mathrm{d}z =$$

$$= \frac{1}{2}\cdot z - 2\int \frac{(z+2)'}{z+2}\,\mathrm{d}z - \frac{1}{2}\int \frac{1}{(z+2)^2}\,\mathrm{d}z = \left| \begin{array}{c} z+2=y \\ \mathrm{d}z = \mathrm{d}y \end{array} \right| =$$

$$= \frac{1}{2}\cdot z - 2\ln|z+2| =$$

$$= \underline{\underline{\sqrt{x^2+4x+5} - \ln\left(x+2+\sqrt{x^2+4x+5}\right)^2 + c}}$$

14. [$x \in (0\,;\,\infty)$] $\underline{\displaystyle\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,\mathrm{d}x} = \displaystyle\int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}\,\mathrm{d}x = \left| \begin{array}{l} \sqrt[6]{x} = z \quad \text{(viz př. 3)} \\ x = z^6 \\ \mathrm{d}x = 6z^5\,\mathrm{d}z \\ x^{\frac{1}{6}} = z \end{array} \right| =$

$$= \int \frac{1}{(z^6)^{\frac{1}{2}}+(z^6)^{\frac{1}{3}}}\cdot 6z^5\,\mathrm{d}z = 6\int \frac{z^5}{z^3+z^2}\,\mathrm{d}z = 6\int \frac{z^2\cdot\{[z^3+1]-1\}}{z^2\cdot(z+1)}\,\mathrm{d}z =$$

$$= 6\int \frac{[(z+1)(z^2-z+1)]}{z+1}\,\mathrm{d}z - 6\int \frac{\{1\}}{z+1}\,\mathrm{d}z =$$

$$= 6\int (z^2-z+1)\,\mathrm{d}z - 6\int \frac{(z+1)'}{z+1}\,\mathrm{d}z = 6\left(\frac{z^3}{3}-\frac{z^2}{2}+z-\ln|z+1|\right)+c =$$

$$= 2z^3-3z^2+6z-\ln(z+1)^6+c = 2\left(x^{\frac{1}{6}}\right)^3 - 3\left(x^{\frac{1}{6}}\right)^2 + 6x^{\frac{1}{6}} - \ln\left(1+x^{\frac{1}{6}}\right)^6 + c =$$

$$= \underline{\underline{2\cdot\sqrt{x} - 3\cdot\sqrt[3]{x} + 6\cdot\sqrt[6]{x} - \ln\left(1+\sqrt[6]{x}\right)^6 + c}}$$

15. $[x \in \mathbb{R} \setminus \{-1\}]$

$$\int \frac{\sqrt[3]{3x+4}}{x+\sqrt[3]{3x+4}}\,\mathrm{d}x = \left| \begin{array}{c} \sqrt[3]{3x+4}=u \\ 3x+4=u^3 \\ x=\dfrac{u^3-4}{3} \\ \mathrm{d}x = u^2\,\mathrm{d}u \end{array} \right| =$$

$$= \int \frac{u}{\frac{u^3-4}{3}+u}\cdot u^2\,\mathrm{d}u = \int \frac{3u^3}{u^3+3u-4}\,\mathrm{d}u = \int \frac{3[(u^3+3u-4)-3u+4]}{u^3+3u-4}\,\mathrm{d}u =$$

$$= \int \left\{ \frac{3(u^3+3u-4)}{u^3+3u-4} + \frac{3[-3u+4]}{(u-1)(u^2+u+4)} \right\} \mathrm{d}u =$$

$$= 3\int \mathrm{d}u + \int \frac{-9u+12}{(u-1)(u^2+u+4)}\,\mathrm{d}u = 3\int \mathrm{d}u + \int \left(\frac{A}{u-1} + \frac{Bu+C}{u^2+u+4} \right) \mathrm{d}u =$$

$$\left| \begin{array}{lcl} & -9u+12 = A(u^2+u+4)+(Bu+C)(u-1) & \\ u=1: & 3 = 6A+0 & A=\frac{1}{2} \\ u=0: & 12 = 4A-C & C=-10 \\ u^2: & 0 = A+B & B=\frac{-1}{2} \end{array} \right|$$

$$= 3u + \int \frac{\frac{1}{2}}{u-1}\,\mathrm{d}u + \int \frac{-\frac{1}{2}u-10}{u^2+u+4}\,\mathrm{d}u =$$

$$= 3u + \frac{1}{2}\int \frac{(u-1)'}{u-1}\,\mathrm{d}u + \int \frac{\frac{-1}{4}[(2u+1)-1]-10}{u^2+u+4}\,\mathrm{d}u =$$

$$= 3u + \frac{1}{2}\ln|u-1| - \frac{1}{4}\int \frac{2u+1}{u^2+u+4}\,\mathrm{d}u + \int \frac{\frac{1}{4}-10}{u^2+u+4}\,\mathrm{d}u =$$

$$= 3u + \frac{1}{4}\ln(u-1)^2 - \frac{1}{4}\int \frac{(u^2+u+4)'}{u^2+u+4}\,\mathrm{d}u + \int \frac{-\frac{39}{4}}{u^2+u+\frac{1}{4}+\frac{15}{4}}\,\mathrm{d}u =$$

$$= 3u + \frac{1}{4}\ln(u-1)^2 - \frac{1}{4}\ln(u^2+u+4) - \frac{39}{4}\int \frac{1}{\left(u+\frac{1}{2}\right)^2+\frac{15}{4}}\,\mathrm{d}u =$$

$$= 3u + \frac{1}{4}\ln\frac{(u-1)^2}{u^2+u+4} - \frac{39}{4}\int \frac{1}{\frac{15}{4}\left[\left(\frac{2u+1}{2}\right)^2\cdot\frac{4}{15}+1\right]}\,\mathrm{d}u =$$

$$= 3u + \frac{1}{4}\ln\frac{(u-1)^2}{u^2+u+4} - \frac{39}{15}\int \frac{1}{\left(\frac{2u+1}{\sqrt{15}}\right)^2+1}\,\mathrm{d}u = \left| \begin{aligned} \frac{2u+1}{\sqrt{15}} &= v \\ 2u+1 &= v\sqrt{15} \\ u &= \frac{v\sqrt{15}-1}{2} \\ \mathrm{d}u &= \frac{\sqrt{15}}{2}\,\mathrm{d}v \end{aligned} \right| =$$

$$= 3u + \frac{1}{4}\ln\frac{(u-1)^2}{u^2+u+4} - \frac{39}{15}\int \frac{1}{v^2+1}\cdot\frac{\sqrt{15}}{2}\,\mathrm{d}v =$$

$$= 3u + \frac{1}{4}\ln\frac{(u-1)^2}{u^2+u+4} - \frac{39}{15}\cdot\frac{\sqrt{15}}{2}\operatorname{arctg} v + c =$$

$$= 3u + \frac{1}{4}\ln\frac{(u-1)^2}{u^2+u+4} - \frac{39}{2\sqrt{15}}\operatorname{arctg}\frac{2u+1}{\sqrt{15}} + c =$$

$$= 3\cdot\sqrt[3]{3x+4} + \frac{1}{4}\ln\frac{\left(\sqrt[3]{3x+4}-1\right)^2}{\sqrt[3]{(3x+4)^2}+\sqrt[3]{3x+4}+4} - \frac{39}{2\cdot\sqrt{15}}\operatorname{arctg}\frac{2\left(\sqrt[3]{3x+4}\right)+1}{\sqrt{15}} + c$$

16. [$x \in \langle -1\,;\,0) \cup (0;\,\infty)$] $$\int \frac{\sqrt{x+1}}{x}\,\mathrm{d}x = \left| \begin{aligned} \sqrt{x+1} &= v \\ x+1 &= v^2 \\ x &= v^2-1 \\ \mathrm{d}x &= 2v\,\mathrm{d}v \end{aligned} \right| = \int \frac{v}{v^2-1}\cdot 2v\,\mathrm{d}v =$$

$$= \int \frac{2v^2}{v^2-1}\,\mathrm{d}v = \int \frac{2[(v^2-1)+1]}{v^2-1}\,\mathrm{d}v = \int \frac{2(v^2-1)}{v^2-1}\,\mathrm{d}v + \int \frac{2[1]}{v^2-1}\,\mathrm{d}v =$$

$$= 2\int \mathrm{d}v + \int \frac{2}{(v+1)(v-1)}\,\mathrm{d}v = 2v + \int \left(\frac{1}{v-1} - \frac{1}{v+1}\right)\mathrm{d}v =$$

$$= 2v + \int \left(\frac{1}{v-1} - \frac{1}{v+1}\right)\mathrm{d}v = 2v + \int \left(\frac{(v-1)'}{v-1} - \frac{(v+1)'}{v+1}\right)\mathrm{d}v =$$

$$= 2v + \ln|v-1| - \ln|v+1| + c = 2v + \ln\left|\frac{v-1}{v+1}\right| + c =$$

$$= 2\sqrt{x+1} + \ln\left|\frac{-1+\sqrt{x+1}}{1+\sqrt{x+1}}\right| + c$$

17. [$x \in (0\,;\,\infty)$] $$\underline{\int \frac{1}{\sqrt{x}\cdot(1+\sqrt[3]{x})}\,\mathrm{d}x} = \int \frac{1}{x^{\frac{1}{2}}+x^{\frac{5}{6}}}\,\mathrm{d}x = \left| \begin{array}{l} \sqrt[6]{x} = z \quad \text{(viz př. 3)} \\ x = z^6 \\ \mathrm{d}x = 6z^5\,\mathrm{d}z \\ x^{\frac{1}{6}} = z \end{array} \right| =$$

$$= \int \frac{1}{(z^6)^{\frac{1}{2}}+(z^6)^{\frac{5}{6}}}\cdot 6z^5\,\mathrm{d}z = \int \frac{6z^5}{z^3+z^5}\,\mathrm{d}z = \int \frac{6z^3\cdot[(z^2+1)-1]}{z^3\cdot(1+z^2)}\,\mathrm{d}z =$$

$$= \int \frac{6(z^2+1)}{1+z^2}\,\mathrm{d}z - \int \frac{6[1]}{1+z^2}\,\mathrm{d}z = 6\int \mathrm{d}z - 6\int \frac{1}{1+z^2}\,\mathrm{d}z = 6z - 6\,\mathrm{arctg}\,z + c =$$

$$= 6x^{\frac{1}{6}} - 6\,\mathrm{arctg}\,x^{\frac{1}{6}} + c = \underline{\underline{6\cdot\sqrt[6]{x} - 6\,\mathrm{arctg}\,\sqrt[6]{x} + c}}$$

18. [$x \in (0\,;\,\infty)$] $$\underline{\int \frac{x-1}{\sqrt{x}}\,\mathrm{d}x} = \int \frac{x}{\sqrt{x}}\,\mathrm{d}x + \int \frac{-1}{\sqrt{x}}\,\mathrm{d}x = \int x^{\frac{1}{2}}\,\mathrm{d}x - \int x^{-\frac{1}{2}}\,\mathrm{d}x =$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = \underline{\underline{\frac{2}{3}\cdot\sqrt{x^3} - 2\cdot\sqrt{x} + c}}$$

19. a) [$x \in \mathbb{R}$] $$\underline{\int \sqrt{x^2+1}\,\mathrm{d}x} = \left| \begin{array}{ll} u = \left(x^2+1\right)^{\frac{1}{2}} & v' = 1 \\ u' = \frac{2x}{2\cdot\sqrt{x^2+1}} & v = x \end{array} \right| =$$

$$= x\cdot\sqrt{x^2+1} - \int \frac{x^2}{\sqrt{x^2+1}}\,\mathrm{d}x = x\cdot\sqrt{x^2+1} - \int \frac{(x^2+1)-1}{\sqrt{x^2+1}}\,\mathrm{d}x =$$

$$= x\cdot\sqrt{x^2+1} - \left[\int \frac{(x^2+1)}{\sqrt{x^2+1}}\,\mathrm{d}x + \int \frac{-1}{\sqrt{x^2+1}}\,\mathrm{d}x\right]$$

$$\Longrightarrow \quad \underline{\int \sqrt{x^2+1}\,\mathrm{d}x} = x\cdot\sqrt{x^2+1} - \underline{\int \sqrt{x^2+1}\,\mathrm{d}x} + \underbrace{\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x}_{\mathcal{J}}$$

$$2\underline{\int \sqrt{x^2+1}\,\mathrm{d}x} = x\cdot\sqrt{x^2+1} + \mathcal{J}$$

$$\underline{\int \sqrt{x^2+1}\,\mathrm{d}x} = \frac{x}{2}\cdot\sqrt{x^2+1} + \frac{1}{2}\cdot\mathcal{J}$$

$$\underline{\int \sqrt{x^2+1}\,\mathrm{d}x} = \underline{\underline{\frac{x}{2}\cdot\sqrt{x^2+1} + \frac{1}{2}\cdot\ln\left(x+\sqrt{x^2+1}\right) + c}}$$

Výpočet integrálu $\mathcal{J} = \int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x =$

$$\left|\begin{aligned} \sqrt{x^2+1} &= w - x \\ x^2+1 &= w^2 - 2xw + x^2 \\ 2xw &= w^2 - 1 \\ x &= \frac{w^2-1}{2w} \\ \mathrm{d}x &= \frac{2w\cdot 2w - (w^2-1)\cdot 2}{(2w)^2}\,\mathrm{d}w \\ \mathrm{d}x &= \frac{w^2+1}{2w^2}\,\mathrm{d}w \end{aligned}\right|$$

$$= \int \frac{1}{w - \frac{w^2-1}{2w}} \cdot \frac{w^2+1}{2w^2}\,\mathrm{d}w = \int \frac{1}{\frac{2w^2-(w^2-1)}{2w}} \cdot \frac{w^2+1}{2w^2}\,\mathrm{d}w =$$

$$= \int \frac{2w}{2w^2 - w^2 + 1} \cdot \frac{w^2+1}{2w^2}\,\mathrm{d}w = \int \frac{1}{w}\,\mathrm{d}w = \ln|w| + c = \ln|\underbrace{x + \underbrace{\sqrt{x^2+1}}_{>|x|}}_{>0}| + c =$$

$$= \ln\left(x + \sqrt{x^2+1}\right) + c$$

19. b) $[x \in \mathbb{R}]$ $\underline{\int \sqrt{x^2+1}\,\mathrm{d}x} =$

$$\left|\begin{aligned} \sqrt{x^2+1} &= u - x \\ x^2+1 &= u^2 - 2xu + x^2 \\ 2xu &= u^2 - 1 \\ x &= \frac{u^2-1}{2u} \\ \mathrm{d}x &= \frac{2u\cdot 2u - (u^2-1)\cdot 2}{(2u)^2}\,\mathrm{d}u \\ \mathrm{d}x &= \frac{2(u^2+1)}{2\cdot 2u^2}\,\mathrm{d}u \end{aligned}\right| =$$

$$= \int \left(u - \frac{u^2-1}{2u}\right) \cdot \frac{u^2+1}{2u^2}\,\mathrm{d}u = \int \frac{2u^2-(u^2-1)}{2u} \cdot \frac{u^2+1}{2u^2}\,\mathrm{d}u =$$

$$= \int \frac{u^2+1}{2u} \cdot \frac{u^2+1}{2u^2}\,\mathrm{d}u = \int \frac{u^4+2u^2+1}{4u^3}\,\mathrm{d}u = \int \left(\frac{u}{4} + \frac{1}{2u} + \frac{1}{4u^3}\right)\mathrm{d}u =$$

$$= \frac{1}{4}\int u\,\mathrm{d}u + \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u + \frac{1}{4}\int u^{-3}\,\mathrm{d}u = \frac{1}{4}\cdot\frac{u^2}{2} + \frac{1}{2}\cdot\ln|u| + \frac{1}{4}\cdot\frac{u^{-2}}{-2} + c =$$

$$= \frac{u^2}{8} + \frac{1}{2}\cdot\ln|u| - \frac{1}{8u^2} + c =$$

$$=\frac{\left(x+\sqrt{x^2+1}\right)^2}{8}+\frac{1}{2}\cdot\ln\left|\underbrace{x+\underbrace{\sqrt{x^2+1}}_{>|x|}}_{>0}\right|-\frac{1}{8\left(x+\sqrt{x^2+1}\right)^2}+c=$$

$$=\frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)+\frac{\left(x+\sqrt{x^2+1}\right)^2}{8}-\frac{1}{8\left(x+\sqrt{x^2+1}\right)^2}\cdot\frac{\left(x-\sqrt{x^2+1}\right)^2}{\left(x-\sqrt{x^2+1}\right)^2}+c=$$

$$=\frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)+\frac{\left(x+\sqrt{x^2+1}\right)^2}{8}-\frac{\left(x-\sqrt{x^2+1}\right)^2}{8\left[\left(x+\sqrt{x^2+1}\right)\cdot\left(x-\sqrt{x^2+1}\right)\right]^2}+c=$$

$$=\frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)+\frac{\left(x+\sqrt{x^2+1}\right)^2}{8}-\frac{\left(x-\sqrt{x^2+1}\right)^2}{8\left[x^2-(x^2+1)\right]^2}+c=$$

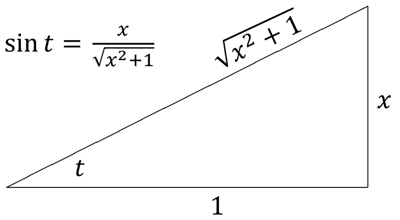
$$=\frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)+\frac{\left(x+\sqrt{x^2+1}\right)^2-\left(x-\sqrt{x^2+1}\right)^2}{8}+c=$$

$$=\frac{\ln\left(x+\sqrt{x^2+1}\right)}{2}+\frac{x^2+2x.\sqrt{x^2+1}+(x^2+1)-\left[x^2-2x.\sqrt{x^2+1}+(x^2+1)\right]}{8}+c=$$

$$=\frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)+\frac{4x.\sqrt{x^2+1}}{8}+c=\underline{\underline{\frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)+\frac{x}{2}\cdot\sqrt{x^2+1}+c}}$$

19. c) $[x\in\mathbb{R}]$ $\underline{\int\sqrt{x^2+1}\,\mathrm{d}x}=$

$$\left|\begin{array}{l} x=\operatorname{tg}t \\ \mathrm{d}x=\dfrac{1}{\cos^2 t}\,\mathrm{d}t \\ \operatorname{arctg}x=t \\ \underline{t\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right)} \end{array}\right.\qquad \sin t=\frac{x}{\sqrt{x^2+1}}$$



$$=\int\sqrt{\operatorname{tg}^2 t+1}\cdot\frac{1}{\cos^2 t}\,\mathrm{d}t=\int\sqrt{\frac{\sin^2 t}{\cos^2 t}+1}\cdot\frac{1}{\cos^2 t}\,\mathrm{d}t=$$

$$=\int\sqrt{\frac{\sin^2 t+\cos^2 t}{\cos^2 t}}\cdot\frac{1}{\cos^2 t}\,\mathrm{d}t=\int\sqrt{\frac{1}{\cos^2 t}}\cdot\frac{1}{\cos^2 t}\,\mathrm{d}t=\int\frac{1}{|\cos t|}\cdot\frac{1}{\cos^2 t}\,\mathrm{d}t=$$

$$=\left|\begin{matrix} \underline{t \in \ldots} \\ \cos t > 0 \end{matrix}\right| = \int \frac{1}{\cos t} \cdot \frac{1}{\cos^2 t} \cdot \frac{\cos t}{\cos t}\ \mathrm{d}t = \int \frac{\cos t}{\left(\cos^2 t\right)^2}\ \mathrm{d}t = \int \frac{\cos t}{\left(1 - \sin^2 t\right)^2}\ \mathrm{d}t =$$

$$=\left|\begin{matrix} \sin t = w \\ \cos t\,\mathrm{d}t = \mathrm{d}w \\ \mathrm{d}t = \dfrac{1}{\cos t}\ \mathrm{d}w \end{matrix}\right| = \int \frac{\cos t}{\left(1 - w^2\right)^2} \cdot \frac{1}{\cos t}\ \mathrm{d}w = \int \frac{1}{\left(1 - w^2\right)^2}\ \mathrm{d}w =$$

$$= \int \frac{1}{\left(w^2 - 1\right)^2}\ \mathrm{d}w = \int \frac{1}{\left[(w+1)(w-1)\right]^2}\ \mathrm{d}w =$$

$$= \int \frac{A}{w+1}\ \mathrm{d}w + \int \frac{B}{(w+1)^2}\ \mathrm{d}w + \int \frac{C}{w-1}\ \mathrm{d}w + \int \frac{D}{(w-1)^2}\ \mathrm{d}w =$$

$$\left|\begin{array}{lll} & 1 = A(w+1)(w-1)^2 + B(w-1)^2 + C(w+1)^2(w-1) + D(w+1)^2 & \\ w = 1: & 1 = 4D & D = \frac{1}{4} \\ w = -1: & 1 = 4B & B = \frac{1}{4} \\ w^3: & 0 = A + C \qquad A = -C & \\ w = 0: & 0 = A + B - C + D & \\ & 0 = A + \frac{1}{4} + A + \frac{1}{4} & A = \frac{1}{4} \\ & & C = \frac{-1}{4} \end{array}\right|$$

$$= \int \frac{\frac{1}{4}}{w+1}\ \mathrm{d}w + \int \frac{\frac{1}{4}}{(w+1)^2}\ \mathrm{d}w + \int \frac{\frac{-1}{4}}{w-1}\ \mathrm{d}w + \int \frac{\frac{1}{4}}{(w-1)^2}\ \mathrm{d}w =$$

$$= \frac{1}{4}\int \frac{(w+1)'}{w+1}\ \mathrm{d}w + \frac{1}{4}\int (w+1)^{-2}\,\mathrm{d}w - \frac{1}{4}\int \frac{(w-1)'}{w-1}\ \mathrm{d}w + \frac{1}{4}\int (w-1)^{-2}\,\mathrm{d}w =$$

$$= \frac{1}{4}\ln|w+1| + \frac{1}{4} \cdot \frac{(w+1)^{-1}}{-1} - \frac{1}{4}\ln|w-1| + \frac{1}{4} \cdot \frac{(w-1)^{-1}}{-1} + c =$$

$$= \frac{1}{4}\ln\left|\frac{w+1}{w-1}\right| - \frac{1}{4(w+1)} - \frac{1}{4(w-1)} + c = \frac{1}{4}\ln\left|\frac{w+1}{w-1}\right| - \frac{(w-1)+(w+1)}{4(w+1)(w-1)} + c =$$

$$= \frac{1}{4}\ln\left|\frac{w+1}{w-1}\right| - \frac{2w}{4(w^2-1)} + c = \frac{1}{4}\ln\left|\frac{w+1}{w-1}\right| - \frac{w}{2(w^2-1)} + c =$$

$$= \frac{1}{4}\ln\left|\frac{\sin t + 1}{\sin t - 1}\right| - \frac{\sin t}{2\left(\sin^2 t - 1\right)} + c = \frac{1}{4}\ln\left|\frac{\frac{x}{\sqrt{x^2+1}} + 1}{\frac{x}{\sqrt{x^2+1}} - 1}\right| - \frac{\frac{x}{\sqrt{x^2+1}}}{2\left[\left(\frac{x}{\sqrt{x^2+1}}\right)^2 - 1\right]} + c =$$

$$= \frac{1}{4}\ln\left|\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}\cdot\frac{\sqrt{x^2+1}}{x-\sqrt{x^2+1}}\right| - \frac{\frac{x}{\sqrt{x^2+1}}}{2\left[\frac{x^2}{x^2+1}-1\right]} + c =$$

$$= \frac{1}{4}\ln\left|\frac{x+\sqrt{x^2+1}}{x-\sqrt{x^2+1}}\right| - \frac{\frac{x}{\sqrt{x^2+1}}}{2\left[\frac{x^2-(x^2+1)}{x^2+1}\right]} + c =$$

$$= \frac{1}{4}\ln\left|\frac{x+\sqrt{x^2+1}}{x-\sqrt{x^2+1}}\cdot\frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2+1}}\right| - \frac{x}{\sqrt{x^2+1}}\cdot\frac{x^2+1}{-2} + c =$$

$$= \frac{1}{4}\ln\left|\frac{\left(x+\sqrt{x^2+1}\right)^2}{x^2-(x^2+1)}\right| + \frac{x}{2}\cdot\sqrt{x^2+1} + c = \frac{1}{4}\ln\frac{\left(x+\sqrt{x^2+1}\right)^2}{|-1|} + \frac{x}{2}\cdot\sqrt{x^2+1} + c =$$

$$= \underline{\underline{\frac{1}{2}\cdot\ln\left(x+\sqrt{x^2+1}\right) + \frac{x}{2}\cdot\sqrt{x^2+1} + c}}$$

19. d) $[x \in \mathbb{R}]$ $\underline{\int\sqrt{x^2+1}\,\mathrm{d}x} = \left|\begin{array}{c} x = \frac{1}{2}\cdot(\mathrm{e}^y-\mathrm{e}^{-y}) \\ \mathrm{d}x = \frac{1}{2}\cdot(\mathrm{e}^y+\mathrm{e}^{-y})\ \mathrm{d}y \end{array}\right| =$

$$= \int\sqrt{\left[\frac{1}{2}\cdot(\mathrm{e}^y-\mathrm{e}^{-y})\right]^2+1}\cdot\frac{1}{2}\cdot(\mathrm{e}^y+\mathrm{e}^{-y})\ \mathrm{d}y =$$

$$= \int\sqrt{\frac{1}{4}\cdot(\mathrm{e}^{2y}-2\underbrace{\mathrm{e}^y\mathrm{e}^{-y}}_{\mathrm{e}^0=1}+\mathrm{e}^{-2y}+4)}\cdot\frac{1}{2}\cdot(\mathrm{e}^y+\mathrm{e}^{-y})\ \mathrm{d}y =$$

$$= \int\sqrt{\frac{1}{4}\cdot(\mathrm{e}^{2y}+2+\mathrm{e}^{-2y})}\cdot\frac{1}{2}\cdot(\mathrm{e}^y+\mathrm{e}^{-y})\,\mathrm{d}y =$$

$$= \int\sqrt{\left[\frac{1}{2}\cdot(\mathrm{e}^y+\mathrm{e}^{-y})\right]^2}\cdot\frac{1}{2}\cdot(\mathrm{e}^y+\mathrm{e}^{-y})\,\mathrm{d}y = \frac{1}{4}\int(\mathrm{e}^y+\mathrm{e}^{-y})^2\,\mathrm{d}y =$$

$$= \frac{1}{4}\int(\mathrm{e}^{2y}+2+\mathrm{e}^{-2y})\,\mathrm{d}y = \frac{1}{4}\cdot\left(\frac{\mathrm{e}^{2y}}{2}+2y+\frac{\mathrm{e}^{-2y}}{-2}\right)+c =$$

Jestliže: $x = \frac{1}{2}\cdot(\mathrm{e}^y-\mathrm{e}^{-y})$ pak: $y = \ln\left(x+\sqrt{x^2+1}\right)$, protože:

$$x=\frac{\mathrm{e}^{\ln\left(x+\sqrt{x^2+1}\right)}-\mathrm{e}^{-\ln\left(x+\sqrt{x^2+1}\right)}}{2}=\frac{1}{2}\cdot\left[\mathrm{e}^{\ln\left(x+\sqrt{x^2+1}\right)}-\mathrm{e}^{\ln\left(x+\sqrt{x^2+1}\right)^{-1}}\right]=$$

$$=\frac{1}{2}\cdot\left(x+\sqrt{x^2+1}-\frac{1}{x+\sqrt{x^2+1}}\right)=$$

$$=\frac{1}{2}\cdot\left(x+\sqrt{x^2+1}-\frac{1}{x+\sqrt{x^2+1}}\cdot\frac{x-\sqrt{x^2+1}}{x-\sqrt{x^2+1}}\right)=$$

$$=\frac{1}{2}\cdot\left[x+\sqrt{x^2+1}-\frac{x-\sqrt{x^2+1}}{x^2-(x^2+1)}\right]=\frac{1}{2}\cdot\left(x+\sqrt{x^2+1}-\frac{x-\sqrt{x^2+1}}{-1}\right)=x$$

$$=\frac{1}{4}\cdot\left[\frac{\mathrm{e}^{2\ln\left(x+\sqrt{x^2+1}\right)}}{2}+2\ln\left(x+\sqrt{x^2+1}\right)+\frac{\mathrm{e}^{-2\ln\left(x+\sqrt{x^2+1}\right)}}{-2}\right]+c=$$

$$=\frac{1}{8}\cdot\left(x+\sqrt{x^2+1}\right)^2+\frac{1}{2}\cdot\ln\left(x+\sqrt{x^2+1}\right)+\frac{1}{-8\cdot\left(x+\sqrt{x^2+1}\right)^2}+c=$$

$$=\frac{1}{8}\cdot\left[\left(x+\sqrt{x^2+1}\right)^2-\frac{1}{\left(x+\sqrt{x^2+1}\right)^2}\cdot\frac{\left(x-\sqrt{x^2+1}\right)^2}{\left(x-\sqrt{x^2+1}\right)^2}\right]+$$

$$+\frac{1}{2}\cdot\ln\left(x+\sqrt{x^2+1}\right)+c=$$

$$=\frac{1}{8}\cdot\left\{\left(x+\sqrt{x^2+1}\right)^2-\frac{\left(x-\sqrt{x^2+1}\right)^2}{[x^2-(x^2+1)]^2}\right\}+\frac{1}{2}\cdot\ln\left(x+\sqrt{x^2+1}\right)+c=$$

$$=\frac{1}{8}\cdot\left[x^2+2x\cdot\sqrt{x^2+1}+(x^2+1)-\frac{x^2-2x\cdot\sqrt{x^2+1}+(x^2+1)}{(-1)^2}\right]+$$

$$+\frac{1}{2}\cdot\ln\left(x+\sqrt{x^2+1}\right)+c=$$

$$=\frac{1}{8}\cdot 4x\cdot\sqrt{x^2+1}+\frac{1}{2}\cdot\ln\left(x+\sqrt{x^2+1}\right)+c=$$

$$=\underline{\underline{\frac{x}{2}\cdot\sqrt{x^2+1}+\frac{1}{2}\cdot\ln\left(x+\sqrt{x^2+1}\right)+c}}$$

19. e) $[x \in \mathbb{R}]$ $\int \sqrt{x^2+1}\,\mathrm{d}x =$

$$\left|\begin{aligned} \sqrt{x^2+1} &= xw+1 \\ x^2+1 &= x^2w^2+2xw+1 \\ 0 &= x^2w^2+2xw-x^2 \\ 0 &= x\cdot(x\cdot w^2+2w-x) \\ 0 &= (x\cdot w^2+2w-x) \\ -2w &= x(\cdot w^2-1) \\ x &= \frac{-2w}{w^2-1} \\ \mathrm{d}x &= \frac{-2(w^2-1)-(-2w)2w}{(w^2-1)^2}\,\mathrm{d}w \\ \mathrm{d}x &= \frac{2w^2+2}{(w^2-1)^2}\,\mathrm{d}w \end{aligned}\right|$$

$$= \int\left(\frac{-2w}{w^2-1}\cdot w+1\right)\cdot\frac{2w^2+2}{(w^2-1)^2}\,\mathrm{d}w = \int\frac{-2w^2+(w^2-1)}{w^2-1}\cdot\frac{2w^2+2}{(w^2-1)^2}\,\mathrm{d}w =$$

$$= \int\frac{(-w^2-1)\cdot(2w^2+2)}{(w^2-1)^3}\,\mathrm{d}w = \int\frac{-2w^4-2w^2-2w^2-2}{[(w+1)(w-1)]^3}\,\mathrm{d}w =$$

$$= \int\left[\frac{A}{w+1}+\frac{B}{(w+1)^2}+\frac{C}{(w+1)^3}+\frac{D}{w-1}+\frac{E}{(w-1)^2}+\frac{F}{(w-1)^3}\right]\mathrm{d}w =$$

$$-2w^4-4w^2-2 =$$
$$= A(w+1)^2(w-1)^3+B(w+1)(w-1)^3+C(w-1)^3+D(w+1)^3(w-1)^2+E(w+1)^3(w-1)+F(w+1)^3$$

$w=1:$	$-8=8F$		$F=-1$
$w=-1:$	$-8=-8C$		$C=1$
$w^5:$	$0=A+D$	$-A=D$	(1)
$w=0:$	$-2=-A-B-C+D-E+F$		
	$0=-B+2D-E$		(2)
$w=2:$	$-50=9A+3B+C+27D+27E+27F$		
	$-24=+3B+18D+27E$		(3)
$w=-2:$	$-50=-27A+27B-27C-9D+3E-F$		
	$-24=+27B+18D+3E$		(4)

Rovnice (2); (3) a (4) řešíme Gaussovou eliminační metodou (sloupce B; D; E):

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 3 & 18 & 27 & -24 \\ 27 & 18 & 3 & -24 \end{array}\right] \begin{array}{c}(-3)(-27)\\ \downarrow \quad | \\ \quad\ \downarrow\end{array} \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 24 & 24 & -24 \\ 0 & 72 & -24 & -24 \end{array}\right] \begin{array}{c} \\ (-3) \\ \downarrow \end{array} \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 24 & 24 & -24 \\ 0 & 0 & -96 & 48 \end{array}\right]$$

a dostáváme: $E=\frac{-1}{2}$; $D=\frac{-1}{2}$; $B=\frac{-1}{2}$; $A=\frac{1}{2}$. Tedy:

$$= \int \left[\frac{\frac{1}{2}}{w+1} + \frac{\frac{-1}{2}}{(w+1)^2} + \frac{1}{(w+1)^3} + \frac{\frac{-1}{2}}{w-1} + \frac{\frac{-1}{2}}{(w-1)^2} + \frac{-1}{(w-1)^3} \right] \mathrm{d}w =$$

$$= \frac{1}{2} \int \frac{(w+1)'}{w+1} \, \mathrm{d}w - \frac{1}{2} \int (w+1)^{-2} \, \mathrm{d}w \quad \left| \, w+1 = u \, \right| \quad + \int (w+1)^{-3} \, \mathrm{d}w -$$

$$- \frac{1}{2} \int \frac{(w-1)'}{w-1} \, \mathrm{d}w - \frac{1}{2} \int (w-1)^{-2} \, \mathrm{d}w \quad \left| \, w-1 = v \, \right| \quad - \int (w-1)^{-3} \, \mathrm{d}w =$$

$$= \ldots = \underline{\underline{\frac{1}{2} \cdot \ln\left(x + \sqrt{x^2+1}\right) + \frac{x}{2} \cdot \sqrt{x^2+1} + c}}$$ [3]

20. [$x \in (0\,;\,1) \cup (1\,;\,\infty)$] $\quad \underline{\displaystyle\int \frac{1}{\sqrt[3]{x} \cdot (1-\sqrt{x})} \, \mathrm{d}x} = \displaystyle\int \frac{1}{x^{\frac{1}{3}} - x^{\frac{1}{3}+\frac{1}{2}}} \, \mathrm{d}x =$

$$= \int \frac{1}{x^{\frac{1}{3}} - x^{\frac{5}{6}}} \, \mathrm{d}x = \left| \begin{array}{l} \sqrt[6]{x} = y \quad \text{(viz př. 3)} \\ x = y^6 \\ \mathrm{d}x = 6y^5 \, \mathrm{d}y \\ x^{\frac{1}{6}} = y \end{array} \right| = \int \frac{1}{(y^6)^{\frac{1}{3}} - (y^6)^{\frac{5}{6}}} \cdot 6y^5 \, \mathrm{d}y =$$

$$= \int \frac{6y^5}{y^2 - y^5} \, \mathrm{d}y = \int \frac{6y^2 \cdot y^3}{-y^2 \cdot (y^3-1)} \, \mathrm{d}y = \int \frac{-6[(y^3-1)+1]}{y^3-1} \, \mathrm{d}y =$$

$$= \int \frac{-6(y^3-1)}{y^3-1} \, \mathrm{d}y + \int \frac{-6[1]}{(y-1)(y^2+y+1)} \, \mathrm{d}y =$$

$$= -6 \int \mathrm{d}y + \int \frac{A}{y-1} \, \mathrm{d}y + \int \frac{By+C}{y^2+y+1} \, \mathrm{d}y =$$

$$\left| \begin{array}{lll} & -6 = A(y^2+y+1) + (By+C)(y-1) & \\ y=1: & -6 = 3A + 0 & A = -2 \\ y=0: & -6 = A - C & C = 4 \\ y^2: & 0 = A + B & B = 2 \end{array} \right|$$

$$= -6 \int \mathrm{d}y + \int \frac{-2}{y-1} \, \mathrm{d}y + \int \frac{2y+4}{y^2+y+1} \, \mathrm{d}y =$$

[3] Tento výsledek (za použití funkce *logarc*) dává webová stránka
`http://um.mendelu.cz/maw-html/index.php?lang=cs&form=integral&function=sqrt(x^2%2B1)`

$$= -6y - 2\int \frac{(y-1)'}{y-1}\, \mathrm{d}y + \int \frac{(2y+1)+3}{y^2+y+1}\, \mathrm{d}y =$$

$$= -6y - 2\ln|y-1| + \int \frac{(2y+1)}{y^2+y+1}\, \mathrm{d}y + \int \frac{3}{y^2+y+\frac{1}{4}+\frac{3}{4}}\, \mathrm{d}y =$$

$$= -6y - \ln(y-1)^2 + \int \frac{(y^2+y+1)'}{y^2+y+1}\, \mathrm{d}y + \int \frac{3}{\left(y+\frac{1}{2}\right)^2+\frac{3}{4}}\, \mathrm{d}y =$$

$$= \left| \begin{array}{c} y+\frac{1}{2} = \sqrt{\frac{3}{4}}u \\ \mathrm{d}y = \frac{\sqrt{3}}{2}\, \mathrm{d}u \end{array} \right| = -6y - \ln(y-1)^2 + \ln(y^2+y+1) + \int \frac{3}{\frac{3}{4}u^2+\frac{3}{4}} \cdot \frac{\sqrt{3}}{2}\, \mathrm{d}u =$$

$$= -6y + \ln\frac{y^2+y+1}{(y-1)^2} + \int \frac{3}{\frac{3}{4}(u^2+1)} \cdot \frac{\sqrt{3}}{2}\, \mathrm{d}u =$$

$$= -6y + \ln\frac{y^2+y+1}{(y-1)^2} + \frac{4}{3}\cdot 3 \cdot \frac{\sqrt{3}}{2}\int \frac{1}{1+u^2}\, \mathrm{d}u =$$

$$= -6y + \ln\frac{y^2+y+1}{(y-1)^2} + 2\cdot\sqrt{3}\cdot \operatorname{arctg} u + c =$$

$$= -6y + \ln\frac{y^2+y+1}{(y-1)^2} + 2\cdot\sqrt{3}\cdot \operatorname{arctg}\frac{2y+1}{\sqrt{3}} + c =$$

$$= \underline{\underline{-6\cdot\sqrt[6]{x} + \ln\frac{\sqrt[6]{x^2}+\sqrt[6]{x}+1}{\left(\sqrt[6]{x}-1\right)^2} + 2\cdot\sqrt{3}\cdot \operatorname{arctg}\frac{2\cdot\sqrt[6]{x}+1}{\sqrt{3}} + c}}$$

21. [$x \in (0\,;\,1) \cup (1\,;\,\infty)$] $\underline{\int \frac{1}{\sqrt{x}\cdot(1-\sqrt[3]{x})}\, \mathrm{d}x} = \int \frac{1}{x^{\frac{1}{2}} - x^{\frac{1}{2}+\frac{1}{3}}}\, \mathrm{d}x =$

$$= \int \frac{1}{x^{\frac{1}{2}} - x^{\frac{5}{6}}}\, \mathrm{d}x = \left| \begin{array}{l} \sqrt[6]{x} = z \quad \text{(viz př. 3)} \\ x = z^6 \\ \mathrm{d}x = 6z^5\, \mathrm{d}z \\ x^{\frac{1}{6}} = z \end{array} \right| = \int \frac{1}{(z^6)^{\frac{1}{2}} - (z^6)^{\frac{5}{6}}} \cdot 6z^5\, \mathrm{d}z =$$

$$= \int \frac{6z^5}{z^3 - z^5}\, \mathrm{d}z = \int \frac{6z^3\cdot z^2}{-z^3\cdot(z^2-1)}\, \mathrm{d}z = \int \frac{-6\cdot[(z^2-1)+1]}{z^2-1}\, \mathrm{d}z =$$

$$= \int \frac{-6(z^2-1)}{z^2-1}\,\mathrm{d}z + \int \frac{-6[1]}{z^2-1}\,\mathrm{d}z = -6\int \mathrm{d}z + \int \frac{-6}{(z+1)(z-1)}\,\mathrm{d}z =$$

$$= -6z + \int \frac{3}{z+1}\,\mathrm{d}z - \int \frac{3}{z-1}\,\mathrm{d}z = 3\int \frac{(z+1)'}{z+1}\,\mathrm{d}z - 3\int \frac{(z-1)'}{z-1}\,\mathrm{d}z - 6z =$$

$$= 3\ln|z+1| - 3\ln|z-1| - 6x^{\frac{1}{6}} + c = 3\ln\left|\frac{z+1}{z-1}\right| - 6\cdot\sqrt[6]{x} =$$

$$= \underline{\underline{3\ln\frac{1+\sqrt[6]{x}}{|-1+\sqrt[6]{x}|} - 6\cdot\sqrt[6]{x} + c}}$$

22. [$x \in (0\,;\,\infty)$] $\underline{\int \frac{x+3}{\sqrt{x}}\,\mathrm{d}x} = \int \frac{x}{\sqrt{x}}\,\mathrm{d}x + \int \frac{3}{\sqrt{x}}\,\mathrm{d}x = \int x^{\frac{1}{2}}\,\mathrm{d}x + 3\int x^{-\frac{1}{2}}\,\mathrm{d}x =$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 3\cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}} = \underline{\underline{\frac{2}{3}\cdot\sqrt{x^3} + 6\cdot\sqrt{x} + c}}$$

23. [$x \in (0\,;\,\infty)$] $\underline{\int \frac{x^2+2}{\sqrt{x}}\,\mathrm{d}x} = \int \frac{x^2}{\sqrt{x}}\,\mathrm{d}x + \int \frac{2}{\sqrt{x}}\,\mathrm{d}x = \int x^{\frac{3}{2}}\,\mathrm{d}x + 2\int x^{-\frac{1}{2}}\,\mathrm{d}x =$

$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 2\cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}} = \underline{\underline{\frac{2}{5}\cdot\sqrt{x^5} + 4\cdot\sqrt{x} + c}}$$

24. [$x \in (0\,;\,\infty)$] $\underline{\int \frac{x^2-1}{\sqrt{x}}\,\mathrm{d}x} = \int \frac{x^2}{\sqrt{x}}\,\mathrm{d}x - \int \frac{1}{\sqrt{x}}\,\mathrm{d}x = \int x^{\frac{3}{2}}\,\mathrm{d}x - \int x^{-\frac{1}{2}}\,\mathrm{d}x =$

$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}} = \underline{\underline{\frac{2}{5}\cdot\sqrt{x^5} - 2\cdot\sqrt{x} + c}}$$

25. $[x \in \mathbb{R}]$ $\underline{\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x} = \left| \begin{array}{l} \sqrt{x^2+1} = w - x \\ x^2 + 1 = w^2 - 2xw + x^2 \\ 2xw = w^2 - 1 \\ x = \dfrac{w^2-1}{2w} \\ \mathrm{d}x = \dfrac{2w\cdot 2w - (w^2-1)\cdot 2}{(2w)^2}\,\mathrm{d}w \\ \mathrm{d}x = \dfrac{w^2+1}{2w^2}\,\mathrm{d}w \end{array} \right| =$

$$= \int \frac{1}{w - \frac{w^2-1}{2w}} \cdot \frac{w^2+1}{2w^2}\,\mathrm{d}w = \int \frac{1}{\frac{2w^2-(w^2-1)}{2w}} \cdot \frac{w^2+1}{2w^2}\,\mathrm{d}w =$$

$$= \int \frac{2w}{2w^2 - w^2 + 1} \cdot \frac{w^2+1}{2w^2}\,\mathrm{d}w = \int \frac{1}{w}\,\mathrm{d}w = \ln|w| + c =$$

$$= \ln|\underbrace{x + \underbrace{\sqrt{x^2+1}}_{>|x|}}_{>0}| + c = \underline{\underline{\ln\left(x + \sqrt{x^2+1}\right) + c}}$$