Neurčitý integrál

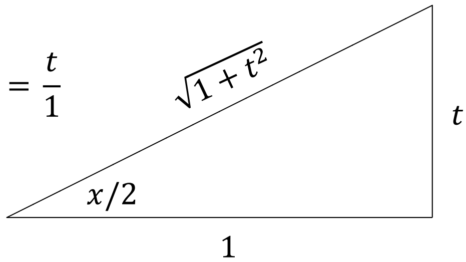
1. Najděte neurčitý integrál $\displaystyle\int \sin x \cos x \,\mathrm{d}x$ $\quad D(f) = \mathbb{R}$

Zde je několik způsobů řešení:

1.1. Substituce $\operatorname{tg}\frac{x}{2} = t$

V pravoúhlém trojúhelníku pro úhel $\frac{x}{2}$ je: $\operatorname{tg}\frac{x}{2} = \frac{t}{1}$

Tím máme stanoveny velikosti obou odvěsen.
Velikost přepony určíme podle Pythagorovy věty.



$$\int \sin x \cos x \,\mathrm{d}x = \left| \begin{array}{ll} \operatorname{tg}\frac{x}{2} = t & x = 2\operatorname{arctg} t \;^{1} \\ \alpha = \frac{x}{2} & \underline{\mathrm{d}x} = \frac{2}{1+t^2}\,\mathrm{d}t \quad \left[\frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2} = t' = \frac{\mathrm{d}t}{\mathrm{d}x}\right] \\ \sin 2\alpha = 2\sin\alpha\cos\alpha & \underline{\sin x} = 2\frac{t}{\sqrt{1+t^2}}\cdot\frac{1}{\sqrt{1+t^2}} = \frac{2\,t}{1+t^2} \\ \cos 2\alpha = \cos^2\alpha - \sin^2\alpha & \underline{\cos x} = \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2 = \frac{1-t^2}{1+t^2} \end{array} \right|$$

$$= \int \frac{2\,t}{1+t^2}\cdot\frac{1-t^2}{1+t^2}\cdot\frac{2}{1+t^2}\,\mathrm{d}t = \int \left[\frac{A\,t+B}{1+t^2} + \frac{C\,t+D}{(1+t^2)^2} + \frac{E\,t+F}{(1+t^2)^3}\right]\mathrm{d}t =$$

$$2t\cdot(1-t^2)\cdot 2 = (At+B)\cdot(1+t^2)^2 + (Ct+D)\cdot(1+t^2) + Et + F$$

$$4t - 4t^3 = (At+B)\cdot(1+2t^2+t^4) + (Ct+D)\cdot(1+t^2) + Et + F$$

$$\begin{array}{llll} t^5: & 0 = A & & \\ t^4: & 0 = B & & \\ t^3: & -4 = 2A + C & \overset{A=0}{\Longrightarrow} & C = -4 \\ t^2: & 0 = 2B + D & \overset{B=0}{\Longrightarrow} & D = 0 \\ t: & 4 = A + C + E & \overset{A=0;C=-4}{\Longrightarrow} & E = 8 \\ t=0: & 0 = B + D + F & \overset{B=0;D=0}{\Longrightarrow} & F = 0 \end{array}$$

$$= \int \frac{-2}{(1+t^2)^2}\,2t\,\mathrm{d}t + \int \frac{4}{(1+t^2)^3}\,2t\,\mathrm{d}t = \left| \begin{array}{l} y = 1+t^2 \\ \mathrm{d}y = 2t\,\mathrm{d}t \end{array} \right| =$$

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[1] Pokud $\operatorname{tg}\frac{x}{2} = t$ a požadujeme vyjádřit x (v závislosti na t), tak funkce $\operatorname{tg}\frac{x}{2}$ není prostá na celém svém definičním oboru.

Proto k ní neexistuje inverzní funkce. Pro každý „kousek" je inverzní funkce jiná (viz pravý sloupec), ale derivace $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{1+t^2}$ je stejná pro libovolné x $(\forall x)$.

$\frac{x}{2} \in \left(\frac{-3\pi}{2}; \frac{-\pi}{2}\right)$: $\frac{x}{2} = \operatorname{arctg} t - \pi$

$\frac{x}{2} \in \left(\frac{-\pi}{2}; \frac{\pi}{2}\right)$: $\frac{x}{2} = \operatorname{arctg} t$

$\frac{x}{2} \in \left(\frac{\pi}{2}; \frac{3\pi}{2}\right)$: $\frac{x}{2} = \operatorname{arctg} t + \pi$

atd.

$$= \int \frac{-2}{y^2}\ \mathrm{d}y + \int \frac{4}{y^3}\ \mathrm{d}y = \int -2y^{-2}\,\mathrm{d}y + \int 4y^{-3}\,\mathrm{d}y = -2\frac{y^{-1}}{-1} + 4\frac{y^{-2}}{-2} + c_1 =$$

$$= \frac{2}{y} - \frac{2}{y^2} + c_1 = \frac{2y-2}{y^2} + c_1 = \frac{2(1+t^2)-2}{(1+t^2)^2} + c_1 = \frac{2t^2}{(1+t^2)^2} + c_1 = \frac{2\,\mathrm{tg}^2\frac{x}{2}}{\left(1+\mathrm{tg}^2\frac{x}{2}\right)^2} + c_1 =$$

$$= \frac{2\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}{\left(1+\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\right)^2} + c_1 = 2\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} \cdot \left(\frac{\cos^2\frac{x}{2}}{\underbrace{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}_{1}}\right)^2 + c_1 = 2\sin^2\frac{x}{2}\cdot\cos^2\frac{x}{2} + c_1 =$$

$$= \frac{1}{2}\left(4\sin^2\frac{x}{2}\cdot\cos^2\frac{x}{2}\right) + c_1 = \frac{1}{2}\left(2\sin\frac{x}{2}\cdot\cos\frac{x}{2}\right)^2 + c_1 = \underline{\underline{\frac{1}{2}\sin^2 x + c_1}}$$

1.2. Substituce $\quad \mathrm{tg}\,x = t$

V pravoúhlém trojúhelníku pro úhel x je: $\quad \mathrm{tg}\,x = \frac{t}{1}$

Tím máme stanoveny velikosti obou odvěsen.
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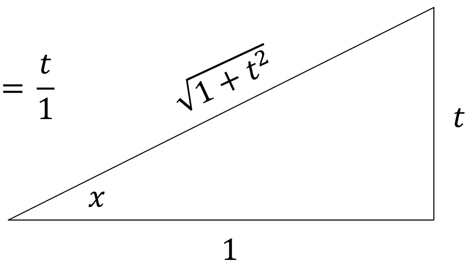

$$\int \sin x \cos x\,\mathrm{d}x = \left| \begin{matrix} t = \mathrm{tg}\,x & \\ \mathrm{d}t = \frac{1}{\cos^2 x}\,\mathrm{d}x & \underline{\sin x} = \frac{t}{\sqrt{1+t^2}} \\ \underline{\mathrm{d}x} = \frac{1}{1+t^2}\,\mathrm{d}t & \underline{\cos x} = \frac{1}{\sqrt{1+t^2}} \end{matrix} \right| = \int \frac{t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}} \cdot \frac{1}{1+t^2}\ \mathrm{d}t =$$

$$= \int \frac{t}{(1+t^2)^2}\ \mathrm{d}t = \left| \begin{matrix} 1+t^2 = z \\ 2t\,\mathrm{d}t = \mathrm{d}z \\ \mathrm{d}t = \frac{1}{2t}\,\mathrm{d}z \end{matrix} \right| = \int \frac{t}{z^2}\cdot\frac{1}{2t}\,\mathrm{d}z = \frac{1}{2}\int z^{-2}\,\mathrm{d}z = \frac{1}{2}\cdot\frac{z^{-1}}{-1} + c_2 =$$

$$= -\frac{1}{2z} + c_2 = -\frac{1}{2(1+t^2)} + c_2 = -\frac{1}{2(1+\mathrm{tg}^2 x)} + c_2 = -\frac{1}{2\left(1+\frac{\sin^2 x}{\cos^2 x}\right)} + c_2 =$$

$$= -\frac{\cos^2 x}{2(\underbrace{\cos^2 x + \sin^2 x}_{1})} + c_2 = \left| c_2 = \frac{1}{2} + c_1 \right| = -\frac{1}{2}\cos^2 x + \frac{1}{2}\cdot 1 + c_1 =$$

$$= -\frac{1}{2}\cos^2 x + \frac{1}{2}\cdot(\sin^2 x + \cos^2 x) + c_1 = \underline{\underline{\frac{1}{2}\sin^2 x + c_1}}$$

1.3. Metoda per partes

$$\int \sin x \cos x \,\mathrm{d}x = \begin{vmatrix} u = \sin x & v' = \cos x \\ u' = \cos x & v = \sin x \end{vmatrix} = \sin x \sin x - \int \cos x \sin x \,\mathrm{d}x$$

$$\int \sin x \cos x \,\mathrm{d}x = \sin x \sin x - \int \sin x \cos x \,\mathrm{d}x$$

$$2\int \sin x \cos x \,\mathrm{d}x = \sin^2 x$$

$$\int \sin x \cos x \,\mathrm{d}x = \tfrac{1}{2}\sin^2 x + c_3 = \ |c_3 = c_1| \ = \underline{\underline{\tfrac{1}{2}\sin^2 x + c_1}}$$

1.4. Substituce $\sin x = y$

$$\int \sin x \cos x \,\mathrm{d}x = \int \sin x \cdot \underline{\cos x \,\mathrm{d}x} = \begin{vmatrix} \sin x = y \\ \cos x \,\mathrm{d}x = \mathrm{d}y \end{vmatrix} = \int y \,\mathrm{d}y = \frac{y^2}{2} + c_4 =$$

$$= \frac{\sin^2 x}{2} + c_4 = \ |c_4 = c_1| \ = \underline{\underline{\frac{1}{2}\sin^2 x + c_1}}$$

1.5. Substituce $\cos x = w$

$$\int \sin x \cos x \,\mathrm{d}x = -\int \cos x \cdot \underline{(-\sin x)\,\mathrm{d}x} = \begin{vmatrix} \cos x = w \\ -\sin x \,\mathrm{d}x = \mathrm{d}w \end{vmatrix} = -\int w \,\mathrm{d}w =$$

$$= -\frac{w^2}{2} + c_5 = -\frac{\cos^2 x}{2} + c_5 = \ \left|c_5 = \frac{1}{2} + c_1\right| \ = -\frac{\cos^2 x}{2} + \frac{1}{2} + c_1 =$$

$$= -\frac{\cos^2 x}{2} + \frac{\sin^2 x + \cos^2 x}{2} + c_1 = \underline{\underline{\frac{1}{2}\sin^2 x + c_1}}$$

1.6. Pomocí vzorce $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$\int \sin x \cos x \,\mathrm{d}x = \frac{1}{2}\int 2\sin x \cos x \,\mathrm{d}x = \frac{1}{2}\int \sin 2x \,\mathrm{d}x = \begin{vmatrix} 2x = z \\ 2\,\mathrm{d}x = \mathrm{d}z \\ \mathrm{d}x = \tfrac{1}{2}\mathrm{d}z \end{vmatrix} =$$

$$= \frac{1}{2}\int \sin z \cdot \frac{1}{2}\mathrm{d}z = -\frac{1}{4}\cos z + c_6 = -\frac{1}{4}\cos 2x + c_6 = -\frac{1}{4}(\cos^2 x - \sin^2 x) + c_6 =$$

$$= \ \left|c_6 = \frac{1}{4} + c_1\right| \ = -\frac{1}{4}(\cos^2 x - \sin^2 x) + \frac{1}{4}(\sin^2 x + \cos^2 x) + c_1 = \underline{\underline{\frac{1}{2}\sin^2 x + c_1}}$$

Další příklady na integrování goniometrických funkcí

2. $[x \neq (2k+1)\pi]$ $\quad \underline{\int \frac{1+\sin x}{1+\cos x}\,\mathrm{d}x} = \int \frac{1+\sin x}{\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)+\cos 2\frac{x}{2}}\,\mathrm{d}x =$

$$= \int \frac{1+\sin x}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}\,\mathrm{d}x = \int \frac{1+\sin x}{2\cos^2\frac{x}{2}}\,\mathrm{d}x =$$

$$= \frac{1}{2}\int \frac{1}{\cos^2\frac{x}{2}}\,\mathrm{d}x + \frac{1}{2}\int \frac{\sin 2\frac{x}{2}}{\cos^2\frac{x}{2}}\,\mathrm{d}x = \left| \begin{array}{c} \frac{x}{2}=w \\ x = 2w \\ dx = 2\,\mathrm{d}w \end{array} \right| =$$

$$= \frac{1}{2}\int \frac{2}{\cos^2 w}\,\mathrm{d}w + \frac{1}{2}\int \frac{2\sin 2w}{\cos^2 w}\,\mathrm{d}w =$$

$$= \int \frac{1}{\cos^2 w}\,\mathrm{d}w + \int \frac{2\sin w\cos w}{\cos^2 w}\,\mathrm{d}w = \operatorname{tg} w + 2\int \frac{\sin w}{\cos w}\,\mathrm{d}w =$$

$$= \operatorname{tg}\frac{x}{2} - 2\int \frac{-\sin w}{\cos w}\,\mathrm{d}w = \operatorname{tg}\frac{x}{2} - 2\int \frac{(\cos w)'}{\cos w}\,\mathrm{d}w = \operatorname{tg}\frac{x}{2} - 2\ln|\cos w| + c =$$

$$= \underline{\underline{\operatorname{tg}\frac{x}{2} - \ln\left(\cos\frac{x}{2}\right)^2 + c}}$$

3. $[x \in \mathbb{R}]$ $\quad \underline{\int \cos^5 x \cdot \sin^2 x\,\mathrm{d}x} = \int (\cos^2 x)^2 \cdot \cos x \cdot \sin^2 x\,\mathrm{d}x =$

$$= \int \left(1-\sin^2 x\right)^2 \cdot \sin^2 x \cdot \underline{\cos x\,\mathrm{d}x} = \left| \begin{array}{c} \sin x = y \\ \cos x\,\mathrm{d}x = \mathrm{d}y \end{array} \right| = \int \left(1-y^2\right)^2 y^2\,\mathrm{d}y =$$

$$= \int (1-2y^2+y^4)\cdot y^2\,\mathrm{d}y = \int (y^2-2y^4+y^6)\,\mathrm{d}y = \frac{y^3}{3} - 2\frac{y^5}{5} + \frac{y^7}{7} + c =$$

$$= \underline{\underline{\frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + c}}$$

4. $[x \in \mathbb{R}]$ $\quad \underline{\int \frac{\sin^3 x}{1+\cos^2 x}\,\mathrm{d}x} = \int \frac{(-\sin^2 x)\cdot(-\sin x)}{1+\cos^2 x}\,\mathrm{d}x =$

$$= \int \frac{(\cos^2 x - 1)}{1+\cos^2 x}\underline{(-\sin x)\,\mathrm{d}x} = \left| \begin{array}{c} \cos x = y \\ -\sin x\,\mathrm{d}x = \mathrm{d}y \end{array} \right| = \int \frac{y^2-1}{y^2+1}\,\mathrm{d}y =$$

$$= \int \frac{(y^2+1)-2}{y^2+1}\,\mathrm{d}y = \int \mathrm{d}y - 2\int \frac{1}{y^2+1}\,\mathrm{d}y =$$

$$= y - 2\operatorname{arctg} y + c = \underline{\underline{\cos x - 2\operatorname{arctg}(\cos x) + c}}$$

5. $[x \neq \operatorname{arctg}\frac{-1}{2} + k\pi]$ [2] $\underline{\displaystyle\int \frac{\sin x + 2\cos x}{\cos x + 2\sin x}\,\mathrm{d}x} = \left| \begin{matrix} \operatorname{tg} x = u & \sin x = \frac{u}{\sqrt{u^2+1}} \\ \mathrm{d}x = \frac{1}{1+u^2}\,\mathrm{d}u & \cos x = \frac{1}{\sqrt{u^2+1}} \end{matrix} \right|$ [3] $=$

$$= \int \frac{\frac{u}{\sqrt{u^2+1}} + \frac{2}{\sqrt{u^2+1}}}{\frac{1}{\sqrt{u^2+1}} + \frac{2u}{\sqrt{u^2+1}}} \cdot \frac{1}{1+u^2}\,\mathrm{d}u = \int \frac{u+2}{1+2u} \cdot \frac{1}{1+u^2}\,\mathrm{d}u =$$

$$= \left| \begin{array}{lcr} & \dfrac{u+2}{(1+2u)(1+u^2)} = \dfrac{A}{1+2u} + \dfrac{Bu+C}{1+u^2} & \\ & u+2 = A(1+u^2) + (Bu+C)(1+2u) & \\ u = -\frac{1}{2}: & \frac{3}{2} = \frac{5}{4}A & A = \frac{6}{5} \\ u = 0: & 2 = A + C & C = \frac{4}{5} \\ u^2: & 0 = A + 2B & B = -\frac{3}{5} \end{array} \right| =$$

$$= \int \left(\frac{\frac{6}{5}}{1+2u} + \frac{-\frac{3}{5}u + \frac{4}{5}}{1+u^2} \right) \mathrm{d}u =$$

$$= \frac{3}{5}\int \frac{2}{1+2u}\,\mathrm{d}u - \frac{3}{10}\int \frac{2u}{1+u^2}\,\mathrm{d}u + \frac{4}{5}\int \frac{1}{1+u^2}\,\mathrm{d}u =$$

$$= \frac{3}{5}\int \frac{(1+2u)'}{1+2u}\,\mathrm{d}u - \frac{3}{10}\int \frac{(1+u^2)'}{1+u^2}\,\mathrm{d}u + \frac{4}{5}\operatorname{arctg} u =$$

$$= \frac{3}{5}\ln|1+2u| - \frac{3}{10}\ln(1+u^2) + \frac{4}{5}\operatorname{arctg}(\operatorname{tg} x) + c =$$

$$= \underline{\underline{\frac{3}{10}\ln\frac{(1+2\operatorname{tg} x)^2}{1+\operatorname{tg}^2 x} + \frac{4}{5}x + c}}$$

6. $[x \in \mathbb{R}]$ $\underline{\displaystyle\int \frac{1}{1+3\cos^2 x}\,\mathrm{d}x} = \left| \operatorname{tg} x = v \right|$ [4] $\displaystyle = \int \frac{1}{1+\frac{3}{v^2+1}} \cdot \frac{1}{1+v^2}\,\mathrm{d}v =$

$$= \int \frac{v^2+1}{(v^2+1)+3} \cdot \frac{1}{1+v^2}\,\mathrm{d}v = \int \frac{1}{v^2+4}\,\mathrm{d}v = \left| \begin{matrix} v = \sqrt{4}w \\ \mathrm{d}v = 2\,\mathrm{d}w \end{matrix} \right| = \int \frac{1}{4w^2+4}\,2\,\mathrm{d}w =$$

$$= \int \frac{2}{4(w^2+1)}\,\mathrm{d}w = \frac{1}{2}\operatorname{arctg} w + c = \frac{1}{2}\operatorname{arctg}\frac{v}{2} + c = \underline{\underline{\frac{1}{2}\operatorname{arctg}\left(\frac{\operatorname{tg} x}{2}\right) + c}}$$

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[2] Definiční obor: $\cos x + 2\sin x \neq 0$
$2\sin x \neq -\cos x$
$\frac{\sin x}{\cos x} \neq -\frac{1}{2}$
$\operatorname{tg} x \neq -\frac{1}{2}$

[3] Viz příklad 1.2.

[4] Viz příklad 5.

7. $[x \in \mathbb{R}]$ $\quad \underline{\int \cos^4 x \cdot \sin^3 x \, dx} = \int \cos^4 x \cdot (-\sin^2 x) \cdot (-\sin x) \, dx =$

$$= \int \cos^4 x \cdot (\cos^2 x - 1) \cdot \underline{(-\sin x) \, dx} = \left| \begin{array}{c} \cos x = y \\ -\sin x \, dx = dy \end{array} \right| = \int y^4 (y^2 - 1) \, dy =$$

$$= \int (y^6 - y^4) \, dy = \frac{y^7}{7} - \frac{y^5}{5} + c = \underline{\underline{\frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + c}}$$

8. $[x \in \mathbb{R}]$ $\underline{\int \cos^2 x \cdot \sin^2 x \, dx} = \left| \begin{array}{c} \cos^2 x = \frac{1+\cos 2x}{2} \\ \sin^2 x = \frac{1-\cos 2x}{2} \end{array} \right| = \left| \begin{array}{r} 1 = \sin^2 x + \cos^2 x \\ \cos 2x = \cos^2 x - \sin^2 x \\ \hline 1 + \cos 2x = 2\cos^2 x \end{array} \right| =$

$$= \int \frac{1+\cos 2x}{2} \cdot \frac{1-\cos 2x}{2} \, dx = \int \frac{1-\cos^2 2x}{4} \, dx = \int \frac{1 - \frac{1+\cos 4x}{2}}{4} \, dx =$$

$$= \frac{1}{4} \int \left(\frac{2 - 1 - \cos 4x}{2} \right) dx = \frac{1}{8} \int (1 - \cos 4x) \, dx = \left| \begin{array}{c} 4x = v \\ 4 \, dx = dv \\ dx = \frac{1}{4} dv \end{array} \right| =$$

$$= \frac{1}{8} x - \frac{1}{8} \int \cos v \cdot \frac{1}{4} \, dv = \frac{1}{8} x - \frac{1}{32} \sin v + c = \underline{\underline{\frac{1}{8} x - \frac{1}{32} \sin 4x + c}}$$

9. $[x \in \mathbb{R}]$ $\quad \underline{\int \sin^3 x \cdot \cos^2 x \, dx} = \int -\sin^2 x \cdot (-\sin x) \cdot \cos^2 x \, dx =$

$$= \int -(1 - \cos^2 x) \cdot \cos^2 x \cdot \underline{(-\sin x) \, dx} = \left| \begin{array}{c} \cos x = w \\ \underline{-\sin x \, dx} = dw \end{array} \right| = \int (w^2 - 1) w^2 \, dw =$$

$$= \int w^4 \, dw - \int w^2 \, dw = \frac{w^5}{5} - \frac{w^3}{3} + c = \underline{\underline{\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + c}}$$

10. [pro $x \in \left\langle 0 ; \frac{\pi}{2} \right\rangle$] $\quad \underline{\int \sin^3 x \cdot \sqrt{9\cos^4 x \sin^2 x + 9 \sin^4 x \cos^2 x} \, dx} =$

$$= \int \sin^3 x \cdot \sqrt{9 \sin^2 x \cos^2 x \overbrace{(\cos^2 x + \sin^2 x)}^{1}} \, dx = \int \sin^3 x \cdot 3 \sin x \cdot \cos x \, dx =$$

$$= 3 \int \sin^4 x \cdot \underline{\cos x \, dx} = \left| \begin{array}{c} \sin x = v \\ \underline{\cos x \, dx} = dv \end{array} \right| = 3 \int v^4 \, dv = 3 \frac{v^5}{5} + c = \underline{\underline{\frac{3}{5} \sin^5 x + c}}$$

11. $\left[\begin{array}{l}\text{Definičním oborem}\\ \text{se nebudeme zabývat}\end{array}\right]$[5] $\underline{\displaystyle\int \frac{1}{\sin^2 x + 3\sin x\cos x + 2\cos^2 x}\,\mathrm{d}x} =$

$$= \left| \begin{array}{ll} \operatorname{tg} x = u & \sin x = \frac{u}{\sqrt{u^2+1}} \\ \mathrm{d}x = \frac{1}{1+u^2}\,\mathrm{d}u & \cos x = \frac{1}{\sqrt{u^2+1}} \end{array} \right| \text{(viz příklad 1.2.)} =$$

$$= \int \frac{1}{\frac{u^2}{u^2+1} + 3\frac{u}{\sqrt{u^2+1}}\frac{1}{\sqrt{u^2+1}} + 2\frac{1}{u^2+1}} \cdot \frac{1}{1+u^2}\,\mathrm{d}u = \int \frac{u^2+1}{u^2+3u+2} \cdot \frac{1}{1+u^2}\,\mathrm{d}u =$$

$$= \int \frac{1}{(u+1)(u+2)}\,\mathrm{d}u = \int \left(\frac{1}{u+1} - \frac{1}{u+2}\right)\mathrm{d}u =$$

$$= \int \left(\frac{(u+1)'}{u+1} - \frac{(u+2)'}{u+2}\right)\mathrm{d}u = \ln(u+1) - \ln(u+2) + c = \ln\frac{u+1}{u+2} + c =$$

$$= \underline{\underline{\ln\frac{\operatorname{tg} x + 1}{\operatorname{tg} x + 2} + c}}$$

12. $[x \in \mathbb{R}]$ $\underline{\displaystyle\int \frac{2-\sin x}{2+\cos x}\,\mathrm{d}x} = \displaystyle\int \frac{2}{2+\cos x}\,\mathrm{d}x + \int \frac{-\sin x}{2+\cos x}\,\mathrm{d}x =$

$$= \int \frac{2}{2+\cos x}\,\mathrm{d}x + \int \frac{(2+\cos x)'}{2+\cos x}\,\mathrm{d}x = \left| \begin{array}{ll} \operatorname{tg}\frac{x}{2} = t & \sin x = \frac{2t}{1+t^2} \\ \mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2} & \cos x = \frac{1-t^2}{1+t^2} \end{array} \right|^{6} =$$

$$= \int \frac{2}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t + \ln(2+\cos x) = \ln(2+\cos x) + \int \frac{4}{2+2t^2+1-t^2}\,\mathrm{d}t =$$

$$= \ln(2+\cos x) + \int \frac{4}{t^2+3}\,\mathrm{d}t = \left| \begin{array}{l} t = \sqrt{3}v \\ \mathrm{d}t = \sqrt{3}\,\mathrm{d}v \end{array} \right| = \ln(2+\cos x) + \int \frac{4}{3v^2+3} \cdot \sqrt{3}\,\mathrm{d}v =$$

$$= \ln(2+\cos x) + \frac{4\sqrt{3}}{3}\int \frac{1}{v^2+1}\,\mathrm{d}v = \ln(2+\cos x) + \frac{4\sqrt{3}}{3}\operatorname{arctg} v + c =$$

$$= \ln(2+\cos x) + \frac{4\sqrt{3}}{3}\operatorname{arctg}\frac{t}{\sqrt{3}} + c = \underline{\underline{\ln(2+\cos x) + \frac{4\sqrt{3}}{3}\operatorname{arctg}\frac{\operatorname{tg}\frac{x}{2}}{\sqrt{3}} + c}}$$

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[5] Kořeny jmenovatele nepatří do definičního oboru. První odhad hodnoty nějakého kořene jmenovatele provedeme grafickou metodou. Ten potom zpřesníme například metodou bisekce (půlení intervalu) a následně zohledníme periodicitu goniometrických funkcí.

[6] Viz příklad 1.1.

13. $\left[\begin{array}{l}\text{Definičním oborem}\\ \text{se nebudeme zabývat}\end{array}\right]^7$ $\underline{\displaystyle\int \frac{1+2\sin x}{\sin x+3\cos x+3}\,\mathrm{d}x} =$

$$= \left| \begin{array}{ll} \operatorname{tg}\frac{x}{2}=t & \sin x=\frac{2t}{1+t^2} \\ \mathrm{d}x=\frac{2\,\mathrm{d}t}{1+t^2} & \cos x=\frac{1-t^2}{1+t^2} \end{array}\right|^8 = \int \frac{1+2\frac{2t}{1+t^2}}{\frac{2t}{1+t^2}+3\frac{1-t^2}{1+t^2}+3}\cdot\frac{2}{1+t^2}\,\mathrm{d}t =$$

$$= \int \frac{\frac{1+t^2+4t}{1+t^2}}{\frac{2t+3-3t^2+3+3t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,\mathrm{d}t = \int \frac{t^2+4t+1}{1+t^2}\cdot\frac{1+t^2}{2t+6}\cdot\frac{2}{1+t^2}\,\mathrm{d}t =$$

$$= \int \frac{t^2+4t+1}{2(t+3)}\cdot\frac{2}{1+t^2}\,\mathrm{d}t = \int\left(\frac{A}{t+3}+\frac{Bt+C}{1+t^2}\right)\mathrm{d}t =$$

$$t^2+4t+1 = A\cdot(1+t^2)+(Bt+C)\cdot(t+3)$$

$$\begin{array}{llcl} t=-3: & 9-12+1=A\cdot(1+9) & & A=-\frac{1}{5} \\ t=0: & 1=A+3C & \overset{A}{\Longrightarrow} & C=\frac{2}{5} \\ t^2: & 1=A+B & \overset{A}{\Longrightarrow} & B=\frac{6}{5} \end{array}$$

$$= \int \frac{-\frac{1}{5}}{t+3}\,\mathrm{d}t+\int\frac{\frac{6}{5}t+\frac{2}{5}}{1+t^2}\,\mathrm{d}t = -\frac{1}{5}\int\frac{(t+3)'}{t+3}\,\mathrm{d}t+\frac{3}{5}\int\frac{2t}{1+t^2}\,\mathrm{d}t+\frac{2}{5}\int\frac{1}{1+t^2}\,\mathrm{d}t =$$

$$= -\frac{1}{5}\ln|t+3|+\frac{3}{5}\ln(1+t^2)+\frac{2}{5}\operatorname{arctg}t+c = \frac{1}{5}\ln\frac{\left(1+t^2\right)^3}{|t+3|}+\frac{2}{5}\operatorname{arctg}t+c =$$

$$= \underline{\underline{\frac{1}{5}\ln\frac{\left(1+\operatorname{tg}^2\frac{x}{2}\right)^3}{\left|3+\operatorname{tg}\frac{x}{2}\right|}+\frac{2}{5}\operatorname{arctg}\left(\operatorname{tg}\frac{x}{2}\right)+c}}$$

14. $[x\in\mathbb{R}]$ $\underline{\displaystyle\int \sin^3 x\,\mathrm{d}x} = \displaystyle\int \sin^2 x\cdot(\sin x)\,\mathrm{d}x = \int(1-\cos^2 x)\cdot(\sin x)\,\mathrm{d}x =$

$$= \int(\cos^2 x-1)\cdot\underline{(-\sin x)\,\mathrm{d}x} = \left|\begin{array}{c}\cos x=w\\ \underline{(-\sin x)\,\mathrm{d}x}=\mathrm{d}w\end{array}\right| = \int(w^2-1)\,\mathrm{d}w =$$

$$= \frac{1}{3}w^3-w+c = \underline{\underline{\frac{1}{3}\cos^3 x-\cos x+c}}$$

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[7] Viz poznámka 5.

[8] Viz příklad 1.1.

15. $\left[x \neq \frac{k\pi}{2}\right]$ $\underline{\displaystyle\int \frac{1}{\sin^4 x \cdot \cos^4 x}\,\mathrm{d}x} = \left| \begin{matrix} \operatorname{tg} x = u & \sin x = \frac{u}{\sqrt{u^2+1}} \\ \mathrm{d}x = \frac{1}{u^2+1}\,\mathrm{d}u & \cos x = \frac{1}{\sqrt{u^2+1}} \end{matrix} \right|^{9} =$

$$= \int \frac{1}{\left(\frac{u}{\sqrt{u^2+1}}\right)^4 \cdot \left(\frac{1}{\sqrt{u^2+1}}\right)^4} \cdot \frac{1}{u^2+1}\,\mathrm{d}u = \int \frac{(u^2+1)^{\frac{4}{2}} \cdot (u^2+1)^{\frac{4}{2}}}{u^4 \cdot 1^4 \cdot (u^2+1)}\,\mathrm{d}u =$$

$$= \int \frac{(u^2+1)^3}{u^4}\,\mathrm{d}u = \int \frac{u^6 + 3u^4 + 3u^2 + 1}{u^4}\,\mathrm{d}u = \int \left(u^2 + 3 + 3u^{-2} + u^{-4}\right)\mathrm{d}u =$$

$$= \frac{1}{3}u^3 + 3u + 3\frac{u^{-1}}{-1} + \frac{u^{-3}}{-3} + c = \underline{\underline{\frac{1}{3}\operatorname{tg}^3 x + 3\operatorname{tg} x - \frac{3}{\operatorname{tg} x} - \frac{1}{3\operatorname{tg}^3 x} + c}}$$

[9] Viz příklad 1.2.