Funkce více proměnných

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Totální diferenciál vyjadřuje závislost změny hodnoty funkce několika (n) proměnných na malé změně jedné nebo více proměnných směrem od daného bodu.

Jestliže totální diferenciál v daném bodě existuje, říkáme, že funkce v daném bodě má totální diferenciál nebo že je v daném bodě ***diferencovatelná***.

Jestliže má funkce $f(x_1, x_2, \dots, x_n)$ na jistém okolí bodu $\mathcal{X}[x_1, x_2, \dots, x_n]$ spojité všechny parciální derivace, pak má v bodě $\mathcal{X}$ totální diferenciál.

1. V bodě $A = [1; -2]$ napište totální diferenciál funkce

$$f(x, y) = 6xy^2 - 2x^3 - 3y^3 . \qquad [x \in \mathbb{R}\,;\ y \in \mathbb{R}]$$

$f_x(1; -2) = 18 \quad f_x = 6y^2 - 6x^2$
$f_y(1; -2) = -60 \quad f_y = 12xy - 9y^2$

$$\mathrm{d}f(1; -2) = 18 \cdot (x - 1) - 60 \cdot (y + 2)$$

Jestliže má funkce $f(x_1, x_2, \dots, x_n)$ na jistém okolí bodu $\mathcal{X} = [x_1, x_2, \dots, x_n]$ totální diferenciál a zároveň parciální derivace $f_{x_1}, f_{x_2}, \dots, f_{x_n}$ mají na jistém okolí stejného bodu $\mathcal{X}$ také totální diferenciály, pak říkáme, že $f(x_1, x_2, \dots, x_n)$ má v bodě $\mathcal{X}$ totální diferenciál ***druhého řádu*** (stručně jen *druhý diferenciál*).

Druhý diferenciál dostaneme formálně jako diferenciál prvého diferenciálu.

2. V bodě $A = [1; -2]$ napište **druhý** totální diferenciál funkce

$$f(x, y) = 6xy^2 - 2x^3 - 3y^3 . \qquad [x \in \mathbb{R}\,;\ y \in \mathbb{R}]$$

$f_x(1; -2) = 18 \quad f_x = 6y^2 - 6x^2$
$f_y(1; -2) = -60 \quad f_y = 12xy - 9y^2$

$f_{xx}(1; -2) = -12 \quad f_{xx} = -12x$
$f_{xy}(1; -2) = -24 \quad f_{xy} = 12y$
$f_{yx}(1; -2) = -24 \quad f_{yx} = 12y$
$f_{yy}(1; -2) = 48 \quad f_{yy} = 12x - 18y$

$$\mathrm{d}^2 f(1; -2) = -12 \cdot (x - 1) \cdot (x - 1) - 24 \cdot (x - 1) \cdot (y + 2) - 24 \cdot (y + 2) \cdot (x - 1) + \\ + 48 \cdot (y + 2) \cdot (y + 2)$$

$$\mathrm{d}^2 f(1; -2) = -12 \cdot (x - 1)^2 - 48 \cdot (x - 1)(y + 2) + 48 \cdot (y + 2)^2$$

3. V bodě $A = [1; -2]$ napište **třetí** totální diferenciál funkce

$$f(x, y) = 6xy^2 - 2x^3 - 3y^3 . \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$f_x(1; -2) = 18 \quad f_x = 6y^2 - 6x^2$
$f_y(1; -2) = -60 \quad f_y = 12xy - 9y^2$

$f_{xx}(1; -2) = -12 \quad f_{xx} = -12x$
$f_{xy}(1; -2) = -24 \quad f_{xy} = 12y$
$f_{yx}(1; -2) = -24 \quad f_{yx} = 12y$
$f_{yy}(1; -2) = 48 \quad f_{yy} = 12x - 18y$

$f_{xxx}(1; -2) = -12 \quad f_{xxx} = -12$
$f_{xxy}(1; -2) = 0 \quad f_{xxy} = 0$
$f_{xyx}(1; -2) = 0 \quad f_{xyx} = 0$
$f_{xyy}(1; -2) = 12 \quad f_{xyy} = 12$
$f_{yxx}(1; -2) = 0 \quad f_{yxx} = 0$
$f_{yxy}(1; -2) = 12 \quad f_{yxy} = 12$
$f_{yyx}(1; -2) = 12 \quad f_{yyx} = 12$
$f_{yyy}(1; -2) = -18 \quad f_{yyy} = -18$

$\mathrm{d}^3 f(1; -2) = -12 \cdot (x-1) \cdot (x-1) \cdot (x-1) + 0 + 0 + 12 \cdot (x-1) \cdot (y+2) \cdot (y+2) + 0 +$
$+ 12 \cdot (y+2) \cdot (x-1) \cdot (y+2) + 12 \cdot (y+2) \cdot (y+2) \cdot (x-1) - 18 \cdot (y+2) \cdot (y+2) \cdot (y+2)$

$$\mathrm{d}^3 f(1; -2) = -12 \cdot (x-1)^3 + 36 \cdot (x-1) \cdot (y+2)^2 - 18 \cdot (y+2)^3$$

Dále pak

$$0 = \mathrm{d}^4 f(1; -2) = \mathrm{d}^5 f(1; -2) = \mathrm{d}^6 f(1; -2) = \dots$$

protože všechny další parciální derivace jsou rovny nule.

4. V bodě $B = [5; 6]$ napište **druhý** totální diferenciál funkce

$$f(x, y) = x^3 + y^2 - 6xy - 39x + 18y + 4 . \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$f_x(5; 6) = 0 \quad f_x = 3x^2 - 6y - 39$
$f_y(5; 6) = 0 \quad f_y = 2y - 6x + 18$

$f_{xx}(5; 6) = 30 \quad f_{xx} = 6x$
$f_{xy}(5; 6) = -6 \quad f_{xy} = -6$
$f_{yx}(5; 6) = -6 \quad f_{yx} = -6$
$f_{yy}(5; 6) = 2 \quad f_{yy} = 2$

$\mathrm{d}^2 f(5; 6) = 30 \cdot (x-5) \cdot (x-5) - 6 \cdot (x-5) \cdot (y-6) - 6 \cdot (y-6) \cdot (x-5) +$
$+ 2 \cdot (y-6) \cdot (y-6)$

$$\mathrm{d}^2 f(5; 6) = 30(x-5)^2 - 12(x-5)(y-6) + 2(y-6)^2$$

5. V bodě $C = [1; -2]$ napište **třetí** totální diferenciál funkce

$$f(x,y) = x^3 + y^2 - 6xy - 39x + 18y + 4\,. \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$$\begin{array}{ll}
f_x(1;-2) = -24 & f_x = 3x^2 - 6y - 39 \\
f_y(1;-2) = 8 & f_y = 2y - 6x + 18
\end{array}$$

$$\begin{array}{ll}
f_{xx}(1;-2) = 6 & f_{xx} = 6x \\
f_{xy}(1;-2) = -6 & f_{xy} = -6 \\
\hline
f_{yx}(1;-2) = -6 & f_{yx} = -6 \\
f_{yy}(1;-2) = 2 & f_{yy} = 2
\end{array}$$

$$\begin{array}{ll}
f_{xxx}(1;-2) = 6 & f_{xxx} = 6 \\
f_{xxy}(1;-2) = 0 & f_{xxy} = 0 \\
\hline
f_{xyx}(1;-2) = 0 & f_{xyx} = 0 \\
f_{xyy}(1;-2) = 0 & f_{xyy} = 0 \\
\hline
f_{yxx}(1;-2) = 0 & f_{yxx} = 0 \\
f_{yxy}(1;-2) = 0 & f_{yxy} = 0 \\
\hline
f_{yyx}(1;-2) = 0 & f_{yyx} = 0 \\
f_{yyy}(1;-2) = 0 & f_{yyy} = 0
\end{array}$$

$$\mathrm{d}^3 f(1;-2) = 6 \cdot (x-1) \cdot (x-1) \cdot (x-1) + 0 + 0 + 0 + 0 + 0 + 0 + 0$$

$$\mathrm{d}^3 f(1;-2) = 6(x-1)^3$$

6. V bodě $D = [1; 2]$ napište **druhý** totální diferenciál funkce

$$f(x,y) = x^3 + y^2 - 6xy - 39x + 18y + 4\,. \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$$\begin{array}{ll}
f_x(1;2) = -48 & f_x = 3x^2 - 6y - 39 \\
f_y(1;2) = 16 & f_y = 2y - 6x + 18
\end{array}$$

$$\begin{array}{ll}
f_{xx}(1;2) = 6 & f_{xx} = 6x \\
f_{xy}(1;2) = -6 & f_{xy} = -6 \\
\hline
f_{yx}(1;2) = -6 & f_{yx} = -6 \\
f_{yy}(1;2) = 2 & f_{yy} = 2
\end{array}$$

$$\mathrm{d}^2 f(1;2) = 6 \cdot (x-1) \cdot (x-1) - 6 \cdot (x-1) \cdot (y-2) - 6 \cdot (y-2) \cdot (x-1) + \\ + 2 \cdot (y-2) \cdot (y+2)$$

$$\mathrm{d}^2 f(1;2) = 6(x-1)^2 - 12(x-1)(y-2) + 2(y-2)^2$$

7. V bodě $P = [0; 0]$ napište **čtvrtý** totální diferenciál funkce

$$f(x, y) = \mathrm{e}^{x+y}. \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$f_x(0;0) = 1 \quad f_x = \mathrm{e}^{x+y} \cdot 1 = \mathrm{e}^{x+y}$
$f_y(0;0) = 1 \quad f_y = \mathrm{e}^{x+y} \cdot 1 = \mathrm{e}^{x+y}$

$f_{xx}(0;0) = 1 \quad f_{xx} = \mathrm{e}^{x+y} \cdot 1 = \mathrm{e}^{x+y}$
$f_{xy}(0;0) = 1 \quad f_{xy} = \mathrm{e}^{x+y} \cdot 1 = \mathrm{e}^{x+y}$
$f_{yx}(0;0) = 1 \quad f_{yx} = \mathrm{e}^{x+y} \cdot 1 = \mathrm{e}^{x+y}$
$f_{yy}(0;0) = 1 \quad f_{yy} = \mathrm{e}^{x+y} \cdot 1 = \mathrm{e}^{x+y}$

$f_{xxx}(0;0) = 1 \quad f_{xxx} = \mathrm{e}^{x+y}$
$f_{xxy}(0;0) = 1 \quad f_{xxy} = \mathrm{e}^{x+y}$
$f_{xyx}(0;0) = 1 \quad f_{xyx} = \mathrm{e}^{x+y}$
$f_{xyy}(0;0) = 1 \quad f_{xyy} = \mathrm{e}^{x+y}$
$f_{yxx}(0;0) = 1 \quad f_{yxx} = \mathrm{e}^{x+y}$
$f_{yxy}(0;0) = 1 \quad f_{yxy} = \mathrm{e}^{x+y}$
$f_{yyx}(0;0) = 1 \quad f_{yyx} = \mathrm{e}^{x+y}$
$f_{yyy}(0;0) = 1 \quad f_{yyy} = \mathrm{e}^{x+y}$

$f_{xxxx}(0;0) = 1 \quad f_{xxxx} = \mathrm{e}^{x+y}$
$f_{xxxy}(0;0) = 1 \quad f_{xxxy} = \mathrm{e}^{x+y}$
$f_{xxyx}(0;0) = 1 \quad f_{xxyx} = \mathrm{e}^{x+y}$
$f_{xxyy}(0;0) = 1 \quad f_{xxyy} = \mathrm{e}^{x+y}$
$f_{xyxx}(0;0) = 1 \quad f_{xyxx} = \mathrm{e}^{x+y}$
$f_{xyxy}(0;0) = 1 \quad f_{xyxy} = \mathrm{e}^{x+y}$
$f_{xyyx}(0;0) = 1 \quad f_{xyyx} = \mathrm{e}^{x+y}$
$f_{xyyy}(0;0) = 1 \quad f_{xyyy} = \mathrm{e}^{x+y}$
$f_{yxxx}(0;0) = 1 \quad f_{yxxx} = \mathrm{e}^{x+y}$
$f_{yxxy}(0;0) = 1 \quad f_{yxxy} = \mathrm{e}^{x+y}$
$f_{yxyx}(0;0) = 1 \quad f_{yxyx} = \mathrm{e}^{x+y}$
$f_{yxyy}(0;0) = 1 \quad f_{yxyy} = \mathrm{e}^{x+y}$
$f_{yyxx}(0;0) = 1 \quad f_{yyxx} = \mathrm{e}^{x+y}$
$f_{yyxy}(0;0) = 1 \quad f_{yyxy} = \mathrm{e}^{x+y}$
$f_{yyyx}(0;0) = 1 \quad f_{yyyx} = \mathrm{e}^{x+y}$
$f_{yyyy}(0;0) = 1 \quad f_{yyyy} = \mathrm{e}^{x+y}$

$$\begin{aligned}\mathrm{d}^4 f(0;0) &= 1 \cdot x \cdot x \cdot x \cdot x + 1 \cdot x \cdot x \cdot x \cdot y + 1 \cdot x \cdot x \cdot y \cdot x + 1 \cdot x \cdot x \cdot y \cdot y + \\ &+ 1 \cdot x \cdot y \cdot x \cdot x + 1 \cdot x \cdot y \cdot x \cdot y + 1 \cdot x \cdot y \cdot y \cdot x + 1 \cdot x \cdot y \cdot y \cdot y + 1 \cdot y \cdot x \cdot x \cdot x + \\ &+ 1 \cdot y \cdot x \cdot x \cdot y + 1 \cdot y \cdot x \cdot y \cdot x + 1 \cdot y \cdot x \cdot y \cdot y + 1 \cdot y \cdot y \cdot x \cdot x + 1 \cdot y \cdot y \cdot x \cdot y + \\ &+ 1 \cdot y \cdot y \cdot y \cdot x + 1 \cdot y \cdot y \cdot y \cdot y = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

$$\mathrm{d}^4 f(0;0) = (x+y)^4$$

8. V bodě $B = [2;1]$ napište **třetí** totální diferenciál funkce

$$f(x,y) = y^3 + 3x^2 - 3y^2 - 3x^2y + 12xy - 11x - 9y + 9\,. \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$f_x(2;1) = 1 \quad f_x = 6x - 6xy + 12y - 11$
$f_y(2;1) = 0 \quad f_y = 3y^2 - 6y - 3x^2 + 12x - 9$

$f_{xx}(2;1) = 0 \quad f_{xx} = 6 - 6y$
$f_{xy}(2;1) = 0 \quad f_{xy} = -6x + 12$
$f_{yx}(2;1) = 0 \quad f_{yx} = -6x + 12$
$f_{yy}(2;1) = 0 \quad f_{yy} = 6y - 6$

$f_{xxx}(2;1) = 0 \quad f_{xxx} = 0$
$f_{xxy}(2;1) = -6 \quad f_{xxy} = -6$
$f_{xyx}(2;1) = -6 \quad f_{xyx} = -6$
$f_{xyy}(2;1) = 0 \quad f_{xyy} = 0$
$f_{yxx}(2;1) = -6 \quad f_{yxx} = -6$
$f_{yxy}(2;1) = 0 \quad f_{yxy} = 0$
$f_{yyx}(2;1) = 0 \quad f_{yyx} = 0$
$f_{yyy}(2;1) = 6 \quad f_{yyy} = 6$

$$\mathrm{d}^3 f(2;1) = 0 - 6\cdot(x-2)\cdot(x-2)\cdot(y-1) - 6\cdot(x-2)\cdot(y-1)\cdot(x-2) + 0 -$$
$$- 6\cdot(y-1)\cdot(x-2)\cdot(x-2) + 0 + 0 + 6\cdot(y-1)\cdot(y-1)\cdot(y-1)$$

$$\mathrm{d}^3 f(2;1) = 6(y-1)^3 - 18(x-2)^2(y-1)$$

9. V bodě $O = [0;0]$ napište **druhý** totální diferenciál funkce

$$f(x,y) = \sin(x+y)\,. \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$f_x(0;0) = 1 \quad f_x = \cos(x+y)\cdot 1 = \cos(x+y)$
$f_y(0;0) = 1 \quad f_y = \cos(x+y)\cdot 1 = \cos(x+y)$

$f_{xx}(0;0) = 0 \quad f_{xx} = -\sin(x+y)\cdot 1 = -\sin(x+y)$
$f_{xy}(0;0) = 0 \quad f_{xy} = -\sin(x+y)\cdot 1 = -\sin(x+y)$
$f_{yx}(0;0) = 0 \quad f_{yx} = -\sin(x+y)\cdot 1 = -\sin(x+y)$
$f_{yy}(0;0) = 0 \quad f_{yy} = -\sin(x+y)\cdot 1 = -\sin(x+y)$

$$\mathrm{d}^2 f(0;0) = 0\cdot x\cdot x + 0\cdot x\cdot y + 0\cdot y\cdot x + 0\cdot y\cdot y$$

$$\mathrm{d}^2 f(0;0) = 0$$

10. V bodě $A = [1; 2]$ napište **třetí** totální diferenciál funkce

$$f(x, y) = x^3 - 3x^2 + 3y^2 - 3xy^2 + 12xy - 9x - 11y + 9 . \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$f_x(1;2) = 0 \quad f_x = 3x^2 - 6x - 3y^2 + 12y - 9$
$f_y(1;2) = 1 \quad f_y = 6y - 6xy + 12x - 11$

$f_{xx}(1;2) = 0 \quad f_{xx} = 6x - 6$
$f_{xy}(1;2) = 0 \quad f_{xy} = -6y + 12$
$f_{yx}(1;2) = 0 \quad f_{yx} = -6y + 12$
$f_{yy}(1;2) = 0 \quad f_{yy} = 6 - 6x$

$f_{xxx}(1;2) = 6 \quad f_{xxx} = 6$
$f_{xxy}(1;2) = 0 \quad f_{xxy} = 0$
$f_{xyx}(1;2) = 0 \quad f_{xyx} = 0$
$f_{xyy}(1;2) = -6 \quad f_{xyy} = -6$
$f_{yxx}(1;2) = 0 \quad f_{yxx} = 0$
$f_{yxy}(1;2) = -6 \quad f_{yxy} = -6$
$f_{yyx}(1;2) = -6 \quad f_{yyx} = -6$
$f_{yyy}(1;2) = 0 \quad f_{yyy} = 0$

$$\mathrm{d}^3 f(1;2) = 6 \cdot (x-1) \cdot (x-1) \cdot (x-1) + 0 + 0 - 6 \cdot (x-1) \cdot (y-2) \cdot (y-2) + 0 -$$
$$- 6 \cdot (y-2) \cdot (x-1) \cdot (y-2) - 6 \cdot (y-2) \cdot (y-2) \cdot (x-1) + 0$$
$$\mathrm{d}^3 f(1;2) = 6(x-1)^3 - 18(x-1)(y-2)^2$$

11. V bodě $C = [1; -1]$ napište **třetí** totální diferenciál funkce

$$f(x, y) = x^3 + xy^2 - 3x + 2xy + 1 . \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$f_x(1;-1) = -1 \quad f_x = 3x^2 + y^2 - 3 + 2y$
$f_y(1;-1) = 0 \quad f_y = 2xy + 2x$

$f_{xx}(1;-1) = 6 \quad f_{xx} = 6x$
$f_{xy}(1;-1) = 0 \quad f_{xy} = 2y + 2$
$f_{yx}(1;-1) = 0 \quad f_{yx} = 2y + 2$
$f_{yy}(1;-1) = 2 \quad f_{yy} = 2x$

$f_{xxx}(1;-1) = 6 \quad f_{xxx} = 6$
$f_{xxy}(1;-1) = 0 \quad f_{xxy} = 0$
$f_{xyx}(1;-1) = 0 \quad f_{xyx} = 0$
$f_{xyy}(1;-1) = 2 \quad f_{xyy} = 2$
$f_{yxx}(1;-1) = 0 \quad f_{yxx} = 0$
$f_{yxy}(1;-1) = 2 \quad f_{yxy} = 2$
$f_{yyx}(1;-1) = 2 \quad f_{yyx} = 2$
$f_{yyy}(1;-1) = 0 \quad f_{yyy} = 0$

$$\mathrm{d}^3 f(1;-1) = 6 \cdot (x-1) \cdot (x-1) \cdot (x-1) + 0 + 0 + 2 \cdot (x-1) \cdot (y+1) \cdot (y+1) + 0 -$$
$$+ 2 \cdot (y+1) \cdot (x-1) \cdot (y+1) + 2 \cdot (y+1) \cdot (y+1) \cdot (x-1) + 0$$
$$\mathrm{d}^3 f(1;-1) = 6(x-1)^3 + 6(x-1)(y+1)^2$$

12. V bodě $P=[0;0]$ napište **čtvrtý** totální diferenciál funkce

$$f(x,y)=\cos(xy)\,. \qquad [x\in\mathbb{R};\ y\in\mathbb{R}]$$

$f_x(0;0)=0 \quad f_x=-\sin(xy)\cdot y=-y\cdot\sin(xy)$
$f_y(0;0)=0 \quad f_y=-\sin(xy)\cdot x=-x\cdot\sin(xy)$

$f_{xx}(0;0)=0 \quad f_{xx}=-y\cdot\cos(xy)\cdot y=-y^2\cdot\cos(xy)$
$f_{xy}(0;0)=0 \quad f_{xy}=-1\cdot\sin(xy)-y\cdot\cos(xy)\cdot x=-\sin(xy)-xy\cdot\cos(xy)$
$f_{yx}(0;0)=0 \quad f_{yx}=-1\cdot\sin(xy)-x\cdot\cos(xy)\cdot y=-\sin(xy)-xy\cdot\cos(xy)$
$f_{yy}(0;0)=0 \quad f_{yy}=-x\cdot\cos(xy)\cdot x=-x^2\cdot\cos(xy)$

$f_{xxx}(0;0)=0 \quad f_{xxx}=-y^2\cdot[-\sin(xy)\cdot y]=y^3\cdot\sin(xy)$
$f_{xxy}(0;0)=0 \quad f_{xxy}=-2y\cdot\cos(xy)-y^2\cdot[-\sin(xy)\cdot x]=$
$=xy^2\cdot\sin(xy)-2y\cdot\cos(xy)$
$f_{xyx}(0;0)=0 \quad f_{xyx}=-\cos(xy)\cdot y-y\cdot\cos(xy)-xy\cdot[-\sin(xy)\cdot y]=$
$=xy^2\cdot\sin(xy)-2y\cdot\cos(xy)$
$f_{xyy}(0;0)=0 \quad f_{xyy}=-\cos(xy)\cdot x-x\cdot\cos(xy)-xy\cdot[-\sin(xy)\cdot x]=$
$=x^2y\cdot\sin(xy)-2x\cdot\cos(xy)$
$f_{yxx}(0;0)=0 \quad f_{yxx}=-\cos(xy)\cdot y-y\cdot\cos(xy)-xy\cdot[-\sin(xy)\cdot y]=$
$=xy^2\cdot\sin(xy)-2y\cdot\cos(xy)$
$f_{yxy}(0;0)=0 \quad f_{yxy}=-\cos(xy)\cdot x-x\cdot\cos(xy)-xy\cdot[-\sin(xy)\cdot x]=$
$=x^2y\cdot\sin(xy)-2x\cdot\cos(xy)$
$f_{yyx}(0;0)=0 \quad f_{yyx}=-2x\cdot\cos(xy)-x^2\cdot[-\sin(xy)\cdot y]=$
$=x^2y\cdot\sin(xy)-2x\cdot\cos(xy)$
$f_{yyy}(0;0)=0 \quad f_{yyy}=-x^2\cdot[-\sin(xy)\cdot x]=x^3\cdot\sin(xy)$

$f_{xxxx}(0;0)=0 \quad f_{xxxx}=y^3\cdot\cos(xy)\cdot y=y^4\cdot\cos(xy)$
$f_{xxxy}(0;0)=0 \quad f_{xxxy}=3y^2\cdot\sin(xy)+y^3\cdot\cos(xy)\cdot x=$
$=3y^2\cdot\sin(xy)+xy^3\cdot\cos(xy)$
$f_{xxyx}(0;0)=0 \quad f_{xxyx}=y^2\cdot\sin(xy)+xy^2\cdot\cos(xy)\cdot y-2y\cdot[-\sin(xy)\cdot y]=$
$=3y^2\cdot\sin(xy)+xy^3\cdot\cos(xy)$
$f_{xxyy}(0;0)=-2 \quad f_{xxyy}=2xy\cdot\sin(xy)+xy^2\cdot\cos(xy)\cdot x-2\cdot\cos(xy)-$
$-2y\cdot[-\sin(xy)\cdot x]=4xy\cdot\sin(xy)+(x^2y^2-2)\cdot\cos(xy)$
$f_{xyxx}(0;0)=0 \quad f_{xyxx}=y^2\cdot\sin(xy)+xy^2\cdot\cos(xy)\cdot y-2y\cdot[-\sin(xy)\cdot y]=$
$=3y^2\cdot\sin(xy)+xy^3\cdot\cos(xy)$
$f_{xyxy}(0;0)=-2 \quad f_{xyxy}=2xy\cdot\sin(xy)+xy^2\cos(xy)\cdot x-2\cos(xy)-$
$-2y\cdot[-\sin(xy)\cdot x]=4xy\cdot\sin(xy)+(x^2y^2-2)\cdot\cos(xy)$
$f_{xyyx}(0;0)=-2 \quad f_{xyyx}=2xy\cdot\sin(xy)+x^2y\cdot\cos(xy)\cdot y-2\cdot\cos(xy)-$
$-2x\cdot[-\sin(xy)\cdot y]=4xy\cdot\sin(xy)+(x^2y^2-2)\cdot\cos(xy)$
$f_{xyyy}(0;0)=0 \quad f_{xyyy}=x^2\cdot\sin(xy)+x^2y\cdot\cos(xy)\cdot x-2x\cdot[-\sin(xy)\cdot x]=$
$=3x^2\cdot\sin(xy)+x^3y\cdot\cos(xy)$

$f_{yxxx}(0;0) = 0$ $\quad f_{yxxx} = y^2 \cdot \sin(xy) + xy^2 \cdot \cos(xy) \cdot y - 2y \cdot [-\sin(xy) \cdot y] = 3y^2 \cdot \sin(xy) + xy^3 \cdot \cos(xy)$

$f_{yxxy}(0;0) = -2$ $\quad f_{yxxy} = 2xy \cdot \sin(xy) + xy^2 \cdot \cos(xy) \cdot x - 2 \cdot \cos(xy) - 2y \cdot [-\sin(xy) \cdot x] = 4xy \cdot \sin(xy) + (x^2y^2 - 2) \cdot \cos(xy)$

$f_{yxyx}(0;0) = -2$ $\quad f_{yxyx} = 2xy \cdot \sin(xy) + x^2y \cdot \cos(xy) \cdot y - 2 \cdot \cos(xy) - 2x \cdot [-\sin(xy) \cdot y] = 4xy \cdot \sin(xy) + (x^2y^2 - 2) \cdot \cos(xy)$

$f_{yxyy}(0;0) = 0$ $\quad f_{yxyy} = x^2 \cdot \sin(xy) + x^2y \cdot \cos(xy) \cdot x - 2x \cdot [-\sin(xy) \cdot x] = 3x^2 \cdot \sin(xy) + x^3y \cdot \cos(xy)$

$f_{yyxx}(0;0) = -2$ $\quad f_{yyxx} = 2xy \cdot \sin(xy) + x^2y \cdot \cos(xy) \cdot y - 2 \cdot \cos(xy) - 2x \cdot [-\sin(xy) \cdot y] = 4xy \cdot \sin(xy) + (x^2y^2 - 2) \cdot \cos(xy)$

$f_{yyxy}(0;0) = 0$ $\quad f_{yyxy} = x^2 \cdot \sin(xy) + x^2y \cdot \cos(xy) \cdot x - 2x \cdot [-\sin(xy) \cdot x] = 3x^2 \cdot \sin(xy) + x^3y \cdot \cos(xy)$

$f_{yyyx}(0;0) = 0$ $\quad f_{yyyx} = 3x^2 \cdot \sin(xy) + x^3 \cdot \cos(xy) \cdot y = 3x^2 \cdot \sin(xy) + x^3y \cdot \cos(xy)$

$f_{yyyy}(0;0) = 0$ $\quad f_{yyyy} = x^3 \cdot \cos(xy) \cdot x = x^4 \cdot \cos(xy)$

$$\begin{aligned} \mathrm{d}^4 f(0;0) &= 0 \cdot x \cdot x \cdot x \cdot x + 0 \cdot x \cdot x \cdot x \cdot y + 0 \cdot x \cdot x \cdot y \cdot x - 2 \cdot x \cdot x \cdot y \cdot y + \\ &+ 0 \cdot x \cdot y \cdot x \cdot x - 2 \cdot x \cdot y \cdot x \cdot y - 2 \cdot x \cdot y \cdot y \cdot x + 0 \cdot x \cdot y \cdot y \cdot y + 0 \cdot y \cdot x \cdot x \cdot x + \\ &- 2 \cdot y \cdot x \cdot x \cdot y - 2 \cdot y \cdot x \cdot y \cdot x + 0 \cdot y \cdot x \cdot y \cdot y - 2 \cdot y \cdot y \cdot x \cdot x + 0 \cdot y \cdot y \cdot x \cdot y + \\ &+ 0 \cdot y \cdot y \cdot y \cdot x + 0 \cdot y \cdot y \cdot y \cdot y = 6 \cdot (-2x^2y^2) \end{aligned}$$

$$\mathrm{d}^4 f(0;0) = -12x^2y^2$$

13. V bodě $D = [-1;1]$ napište **druhý** totální diferenciál funkce

$$f(x,y) = y^3 + x^2y + 2xy - 3y + 1\,. \qquad [x \in \mathbb{R}\,;\ y \in \mathbb{R}]$$

$f_x(-1;1) = 0$ $\quad f_x = 2xy + 2y$
$f_y(-1;1) = -1$ $\quad f_y = 3y^2 + x^2 + 2x - 3$

$f_{xx}(-1;1) = 2$ $\quad f_{xx} = 2y$
$f_{xy}(-1;1) = 0$ $\quad f_{xy} = 2x + 2$
$f_{yx}(-1;1) = 0$ $\quad f_{yx} = 2x + 2$
$f_{yy}(-1;1) = 6$ $\quad f_{yy} = 6y$

$$\mathrm{d}^2 f(-1;1) = 2 \cdot (x+1) \cdot (x+1) + 0 \cdot (x+1) \cdot (y-1) + 0 \cdot (y-1) \cdot (x+1) + 6 \cdot (y-1) \cdot (y-1)$$

$$\mathrm{d}^2 f(-1;1) = 2(x+1)^2 + 6(y-1)^2$$

14. V bodě $O = [0; 0]$ napište **třetí** totální diferenciál funkce

$$f(x, y) = \mathrm{e}^{x^2+y^2}. \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$f_x(0;0) = 0 \quad f_x = \mathrm{e}^{x^2+y^2} \cdot 2x = 2x \cdot \mathrm{e}^{x^2+y^2}$

$f_y(0;0) = 0 \quad f_y = \mathrm{e}^{x^2+y^2} \cdot 2y = 2y \cdot \mathrm{e}^{x^2+y^2}$

$f_{xx}(0;0) = 2 \quad f_{xx} = 2\mathrm{e}^{x^2+y^2} + 2x\mathrm{e}^{x^2+y^2} \cdot 2x = (2 + 4x^2) \cdot \mathrm{e}^{x^2+y^2}$

$f_{xy}(0;0) = 0 \quad f_{xy} = 2x\mathrm{e}^{x^2+y^2} \cdot 2y = 4xy \cdot \mathrm{e}^{x^2+y^2}$

$f_{yx}(0;0) = 0 \quad f_{yx} = 2y\mathrm{e}^{x^2+y^2} \cdot 2x = 4xy \cdot \mathrm{e}^{x^2+y^2}$

$f_{yy}(0;0) = 2 \quad f_{yy} = 2\mathrm{e}^{x^2+y^2} + 2y\mathrm{e}^{x^2+y^2} \cdot 2y = (2 + 4y^2) \cdot \mathrm{e}^{x^2+y^2}$

$f_{xxx}(0;0) = 0 \quad f_{xxx} = 8x\mathrm{e}^{x^2+y^2} + (2 + 4x^2)\mathrm{e}^{x^2+y^2} \cdot 2x = (12x + 8x^3) \cdot \mathrm{e}^{x^2+y^2}$

$f_{xxy}(0;0) = 0 \quad f_{xxy} = (2 + 4x^2)\mathrm{e}^{x^2+y^2} \cdot 2y = (4y + 8x^2y) \cdot \mathrm{e}^{x^2+y^2}$

$f_{xyx}(0;0) = 0 \quad f_{xyx} = 4y\mathrm{e}^{x^2+y^2} + 4xy\mathrm{e}^{x^2+y^2} \cdot 2x = (4y + 8x^2y) \cdot \mathrm{e}^{x^2+y^2}$

$f_{xyy}(0;0) = 0 \quad f_{xyy} = 4x\mathrm{e}^{x^2+y^2} + 4xy\mathrm{e}^{x^2+y^2} \cdot 2y = (4x + 8xy^2) \cdot \mathrm{e}^{x^2+y^2}$

$f_{yxx}(0;0) = 0 \quad f_{yxx} = 4y\mathrm{e}^{x^2+y^2} + 4xy\mathrm{e}^{x^2+y^2} \cdot 2x = (4y + 8x^2y) \cdot \mathrm{e}^{x^2+y^2}$

$f_{yxy}(0;0) = 0 \quad f_{yxy} = 4x\mathrm{e}^{x^2+y^2} + 4xy\mathrm{e}^{x^2+y^2} \cdot 2y = (4x + 8xy^2) \cdot \mathrm{e}^{x^2+y^2}$

$f_{yyx}(0;0) = 0 \quad f_{yyx} = (2 + 4y^2)\mathrm{e}^{x^2+y^2} \cdot 2x = (4x + 8xy^2) \cdot \mathrm{e}^{x^2+y^2}$

$f_{yyy}(0;0) = 0 \quad f_{yyy} = 8y\mathrm{e}^{x^2+y^2} + (2 + 4y^2)\mathrm{e}^{x^2+y^2} \cdot 2y = (12y + 8y^3) \cdot \mathrm{e}^{x^2+y^2}$

$$\mathrm{d}^3 f(0;0) = 0$$

15. V bodě $P = [0; 0]$ napište **druhý** totální diferenciál funkce

$$f(x, y) = \sin(x^2 + y^2). \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$f_x(0;0) = 0 \quad f_x = \cos(x^2 + y^2) \cdot 2x = 2x \cdot \cos(x^2 + y^2)$

$f_y(0;0) = 0 \quad f_y = \cos(x^2 + y^2) \cdot 2y = 2y \cdot \cos(x^2 + y^2)$

$f_{xx}(0;0) = 2 \quad f_{xx} = 2 \cdot \cos(x^2 + y^2) - 2x \cdot \sin(x^2 + y^2) \cdot 2x =$

$= 2 \cdot \cos(x^2 + y^2) - 4x^2 \cdot \sin(x^2 + y^2)$

$f_{xy}(0;0) = 0 \quad f_{xy} = -2x \cdot \sin(x^2 + y^2) \cdot 2y = -4xy \cdot \sin(x^2 + y^2)$

$f_{yx}(0;0) = 0 \quad f_{yx} = -2y \cdot \sin(x^2 + y^2) \cdot 2x = -4xy \cdot \sin(x^2 + y^2)$

$f_{yy}(0;0) = 2 \quad f_{yy} = 2 \cdot \cos(x^2 + y^2) - 2y \cdot \sin(x^2 + y^2) \cdot 2y =$

$= 2 \cdot \cos(x^2 + y^2) - 4y^2 \cdot \sin(x^2 + y^2)$

$\mathrm{d}^2 f(0;0) = 2 \cdot x^2 + 0 \cdot xy + 0 \cdot yx + 2 \cdot y^2$

$$\mathrm{d}^2 f(0;0) = 2 \cdot (x^2 + y^2)$$