Funkce více proměnných

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Taylorův polynom

Taylorův rozvoj v bodě $\mathcal{A} = [x_0, \dots, z_0]$ pro funkci více (reálných) proměnných (která má spojité parciální derivace) vyjádříme pomocí totálních diferenciálů jako

$$\underline{T_n(x, \dots, z) = f(x_0, \dots, z_0) + \frac{\mathrm{d}f(x_0, \dots, z_0)}{1!} + \frac{\mathrm{d}^2 f(x_0, \dots, z_0)}{2!} + \frac{\mathrm{d}^3 f(x_0, \dots, z_0)}{3!} + \cdots}$$

Taylorova řada se používá k polynomiální aproximaci funkcí, protože platí, že všechny derivace Taylorova polynomu stupně n mají ve středu rozvoje (v bodě $\mathcal{A}[x_0, \dots, z_0]$) stejné funkční hodnoty jako odpovídající derivace funkce f. Tato aproximace je na okolí bodu $\mathcal{A}$ (středu rozvoje) tím přesnější, čím vyšší stupeň polynomu použijeme. Zároveň platí, že se chyba se vzdáleností od středu zvyšuje.

V případě, že má Taylorův polynom střed v počátku, pak se označuje jako ***Maclaurinův*** polynom.

1. V okolí středu $A = [1; -2]$ rozviňte do **Taylorova mnohočlenu** (nultého, prvého, druhého, třetího a čtvrtého stupně) funkci:

$$f(x, y) = 6xy^2 - 2x^3 - 3y^3 . \qquad [x \in \mathbb{R}\,;\ y \in \mathbb{R}]$$

$f(1; -2) = 6 \cdot 1 \cdot (-2)^2 - 2 \cdot 1^3 - 3 \cdot (-2)^3 = 24 - 2 + 24 = 46$

$\underline{\underline{T_0(A) = 46}}$

$f_x(1; -2) = 18 \quad f_x = 6y^2 - 6x^2$
$f_y(1; -2) = -60 \quad f_y = 12xy - 9y^2$

$\mathrm{d}f(1; -2) = 18 \cdot (x - 1) - 60 \cdot (y + 2)$

$\underline{\underline{T_1(A) = 46 + 18 \cdot (x - 1) - 60 \cdot (y + 2)}}$

$f_{xx}(1; -2) = -12 \quad f_{xx} = -12x$
$f_{xy}(1; -2) = -24 \quad f_{xy} = 12y$
$f_{yx}(1; -2) = -24 \quad f_{yx} = 12y$
$f_{yy}(1; -2) = 48 \quad f_{yy} = 12x - 18y$

$\mathrm{d}^2 f(1; -2) = -12 \cdot (x - 1) \cdot (x - 1) - 24 \cdot (x - 1) \cdot (y + 2) - 24 \cdot (y + 2) \cdot (x - 1) +$
$+ 48 \cdot (y + 2) \cdot (y + 2) = -12 \cdot (x - 1)^2 - 48 \cdot (x - 1)(y + 2) + 48 \cdot (y + 2)^2$

$\underline{\underline{T_2(A) = 46 + 18.(x - 1) - 60.(y + 2) - 6.(x - 1)^2 - 24.(x - 1).(y + 2) + 24.(y + 2)^2}}$

$$f_{xxx}(1;-2) = -12 \quad f_{xxx} = -12$$
$$f_{xxy}(1;-2) = 0 \quad f_{xxy} = 0$$
$$f_{xyx}(1;-2) = 0 \quad f_{xyx} = 0$$
$$f_{xyy}(1;-2) = 12 \quad f_{xyy} = 12$$
$$f_{yxx}(1;-2) = 0 \quad f_{yxx} = 0$$
$$f_{yxy}(1;-2) = 12 \quad f_{yxy} = 12$$
$$f_{yyx}(1;-2) = 12 \quad f_{yyx} = 12$$
$$f_{yyy}(1;-2) = -18 \quad f_{yyy} = -18$$

$$\mathrm{d}^3 f(1;-2) = -12 \cdot (x-1) \cdot (x-1) \cdot (x-1) + 0 + 12 \cdot (x-1) \cdot (y+2) \cdot (y+2) +$$
$$+12 \cdot (y+2) \cdot (x-1) \cdot (y+2) + 12 \cdot (y+2) \cdot (y+2) \cdot (x-1) - 18 \cdot (y+2) \cdot (y+2) \cdot (y+2) =$$
$$= -12 \cdot (x-1)^3 + 36 \cdot (x-1) \cdot (y+2)^2 - 18 \cdot (y+2)^3$$

$$T_3(A) = 46 + 18 \cdot (x-1) - 60 \cdot (y+2) - 6 \cdot (x-1)^2 - 24 \cdot (x-1) \cdot (y+2) +$$
$$+ 24 \cdot (y+2)^2 - 2 \cdot (x-1)^3 + 6 \cdot (x-1) \cdot (y+2)^2 - 3 \cdot (y+2)^3$$

$$f_{xxxx} = f_{xxxy} = 0$$
$$f_{xxyx} = f_{xxyy} = 0$$
$$f_{xyxx} = f_{xyxy} = 0$$
$$f_{xyyx} = f_{xyyy} = 0$$
$$f_{yxxx} = f_{yxxy} = 0$$
$$f_{yxyx} = f_{yxyy} = 0$$
$$f_{yyxx} = f_{yyxy} = 0$$
$$f_{yyyx} = f_{yyyy} = 0$$

$$\mathrm{d}^4 f(1;-2) = 0$$

$$T_4(A) = 46 + 18 \cdot (x-1) - 60 \cdot (y+2) - 6 \cdot (x-1)^2 - 24 \cdot (x-1) \cdot (y+2) +$$
$$+ 24 \cdot (y+2)^2 - 2 \cdot (x-1)^3 + 6 \cdot (x-1) \cdot (y+2)^2 - 3 \cdot (y+2)^3 + 0 \qquad = T_3(A)$$

$$\mathrm{d}^5 f(1;-2) = 0$$
$$\vdots$$

$$T_3(A) = T_4(A) = T_5(A) = T_6(A) = \dots$$

2. a) V okolí středu $B = [5; 6]$ rozviňte do **Taylorovy** řady funkci

$$f(x,y) = x^3 + y^2 - 6xy - 39x + 18y + 4 . \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$$f(5;6) = 5^3 + 6^2 - 6\cdot 5\cdot 6 - 39\cdot 5 + 18\cdot 6 + 4 = 125 + 36 - 180 - 195 + 108 + 4 = -102$$

$f_x(5;6) = 0 \quad f_x = 3x^2 - 6y - 39$
$f_y(5;6) = 0 \quad f_y = 2y - 6x + 18$

$$\mathrm{d}f(5;6) = 0\cdot(x-5) + 0\cdot(y-6) = 0$$

$f_{xx}(5;6) = 30 \quad f_{xx} = 6x$
$f_{xy}(5;6) = -6 \quad f_{xy} = -6$
$f_{yx}(5;6) = -6 \quad f_{yx} = -6$
$f_{yy}(5;6) = 2 \quad f_{yy} = 2$

$$\mathrm{d}^2 f(5;6) = 30\cdot(x-5)\cdot(x-5) - 6\cdot(x-5)\cdot(y-6) - 6\cdot(y-6)\cdot(x-5) + \\ + 2\cdot(y-6)\cdot(y-6) = 30(x-5)^2 - 12(x-5)(y-6) + 2(y-6)^2$$

$f_{xxx}(5;6) = 6 \quad f_{xxx} = 6$
$f_{xxy}(5;6) = 0 \quad f_{xxy} = 0$
$f_{xyx}(5;6) = 0 \quad f_{xyx} = 0$
$f_{xyy}(5;6) = 0 \quad f_{xyy} = 0$
$f_{yxx}(5;6) = 0 \quad f_{yxx} = 0$
$f_{yxy}(5;6) = 0 \quad f_{yxy} = 0$
$f_{yyx}(5;6) = 0 \quad f_{yyx} = 0$
$f_{yyy}(5;6) = 0 \quad f_{yyy} = 0$

$$\mathrm{d}^3 f(5;6) = 6\cdot(x-5)\cdot(x-5)\cdot(x-5) + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 6(x-5)^3$$

Všechny další parciální derivace jsou rovny nule,

$$\mathrm{d}^4 f(5;6) = 0;\ \mathrm{d}^5 f(5;6) = 0;\ \mathrm{d}^6 f(5;6) = 0;\ \mathrm{d}^7 f(5;6) = 0;\ \ldots \quad \text{proto}$$

$$\underline{\underline{T(B) =}} \quad f(5;6) + \frac{\mathrm{d}f(5;6)}{1!} + \frac{\mathrm{d}^2 f(5;6)}{2!} + \frac{\mathrm{d}^3 f(5;6)}{3!} =$$
$$\underline{\underline{= -102 + 15(x-5)^2 - 6(x-5)(y-6) + (y-6)^2 + (x-5)^3}}$$

2. b) V okolí středu $C = [1; -2]$ rozviňte do **Taylorovy** řady funkci

$$f(x, y) = x^3 + y^2 - 6xy - 39x + 18y + 4 . \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$$f(1;-2) = 1^3 + (-2)^2 - 6 \cdot 1 \cdot (-2) - 39 \cdot 1 + 18 \cdot (-2) + 4 = 1 + 4 + 12 - 39 - 36 + 4 = -54$$

$$\begin{array}{ll} f_x(1;-2) = -24 & f_x = 3x^2 - 6y - 39 \\ f_y(1;-2) = 8 & f_y = 2y - 6x + 18 \end{array}$$

$$\mathrm{d}f(1;-2) = -24 \cdot (x-1) + 8 \cdot (y+2)$$

$$\begin{array}{ll} f_{xx}(1;-2) = 6 & f_{xx} = 6x \\ f_{xy}(1;-2) = -6 & f_{xy} = -6 \\ \hline f_{yx}(1;-2) = -6 & f_{yx} = -6 \\ f_{yy}(1;-2) = 2 & f_{yy} = 2 \end{array}$$

$$\mathrm{d}^2 f(1;-2) = 6 \cdot (x-1) \cdot (x-1) - 6 \cdot (x-1) \cdot (y+2) - 6 \cdot (y+2) \cdot (x-1) +$$
$$+ 2 \cdot (y+2) \cdot (y+2) = 6(x-1)^2 - 12(x-5)(y+2) + 2(y+2)^2$$

$$\begin{array}{ll} f_{xxx}(1;-2) = 6 & f_{xxx} = 6 \\ f_{xxy}(1;-2) = 0 & f_{xxy} = 0 \\ \hline f_{xyx}(1;-2) = 0 & f_{xyx} = 0 \\ f_{xyy}(1;-2) = 0 & f_{xyy} = 0 \\ \hline f_{yxx}(1;-2) = 0 & f_{yxx} = 0 \\ f_{yxy}(1;-2) = 0 & f_{yxy} = 0 \\ \hline f_{yyx}(1;-2) = 0 & f_{yyx} = 0 \\ f_{yyy}(1;-2) = 0 & f_{yyy} = 0 \end{array}$$

$$\mathrm{d}^3 f(1;-2) = 6 \cdot (x-1) \cdot (x-1) \cdot (x-1) + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 6(x-1)^3$$

Všechny další parciální derivace jsou rovny nule,

$$\mathrm{d}^4 f(1;-2) = 0;\ \mathrm{d}^5 f(1;-2) = 0;\ \mathrm{d}^6 f(1;-2) = 0;\ \mathrm{d}^7 f(1;-2) = 0;\ \ldots \quad \text{proto}$$

$$\underline{\underline{T(C) =}} \quad f(1;-2) + \frac{\mathrm{d}f(1;-2)}{1!} + \frac{\mathrm{d}^2 f(1;-2)}{2!} + \frac{\mathrm{d}^3 f(1;-2)}{3!} =$$

$$\underline{\underline{= -54 - 24(x-1) + 8(y+2) + 3(x-1)^2 - 6(x-1)(y+2) + (y+2)^2 + (x-1)^3}}$$

2. c) V okolí středu $D = [1; 2]$ rozviňte do **Taylorovy** řady funkci

$$f(x, y) = x^3 + y^2 - 6xy - 39x + 18y + 4 . \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$$f(1; 2) = 1^3 + 2^2 - 6 \cdot 1 \cdot 2 - 39 \cdot 1 + 18 \cdot 2 + 4 = 1 + 4 - 12 - 39 + 36 + 4 = -6$$

$$\begin{array}{ll}
f_x(1; 2) = -48 & f_x = 3x^2 - 6y - 39 \\
f_y(1; 2) = \ \ 16 & f_y = 2y - 6x + 18
\end{array}$$

$$\mathrm{d}f(1; 2) = -48 \cdot (x - 1) + 16 \cdot (y - 2)$$

$$\begin{array}{ll}
f_{xx}(1; 2) = \ \ 6 & f_{xx} = 6x \\
f_{xy}(1; 2) = -6 & f_{xy} = -6 \\
\hline
f_{yx}(1; 2) = -6 & f_{yx} = -6 \\
f_{yy}(1; 2) = \ \ 2 & f_{yy} = \ \ 2
\end{array}$$

$$\mathrm{d}^2 f(1; 2) = 6 \cdot (x - 1) \cdot (x - 1) - 6 \cdot (x - 1) \cdot (y - 2) - 6 \cdot (y - 2) \cdot (x - 1) +$$
$$+ 2 \cdot (y - 2) \cdot (y + 2) = 6(x - 1)^2 - 12(x - 5)(y - 2) + 2(y - 2)^2$$

$$\begin{array}{ll}
f_{xxx}(1; 2) = 6 & f_{xxx} = 6 \\
f_{xxy}(1; 2) = 0 & f_{xxy} = 0 \\
\hline
f_{xyx}(1; 2) = 0 & f_{xyx} = 0 \\
f_{xyy}(1; 2) = 0 & f_{xyy} = 0 \\
\hline
f_{yxx}(1; 2) = 0 & f_{yxx} = 0 \\
f_{yxy}(1; 2) = 0 & f_{yxy} = 0 \\
\hline
f_{yyx}(1; 2) = 0 & f_{yyx} = 0 \\
f_{yyy}(1; 2) = 0 & f_{yyy} = 0
\end{array}$$

$$\mathrm{d}^3 f(1; 2) = 6 \cdot (x - 1) \cdot (x - 1) \cdot (x - 1) + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 6(x - 1)^3$$

Všechny další parciální derivace jsou rovny nule,

$$\mathrm{d}^4 f(1; 2) = 0;\ \mathrm{d}^5 f(1; 2) = 0;\ \mathrm{d}^6 f(1; 2) = 0;\ \mathrm{d}^7 f(1; 2) = 0;\ \ldots \quad \text{proto}$$

$$\underline{\underline{T(D) =}} \quad f(1; 2) + \frac{\mathrm{d}f(1; 2)}{1!} + \frac{\mathrm{d}^2 f(1; 2)}{2!} + \frac{\mathrm{d}^3 f(1; 2)}{3!} =$$

$$\underline{\underline{= -6 - 48(x - 1) + 16(y - 2) + 3(x - 1)^2 - 6(x - 1)(y - 2) + (y - 2)^2 + (x - 1)^3}}$$

3. Rozviňte do **Maclaurinovy** řady (⇒ Taylorův rozvoj se středem v počátku $O=[0;0]$) funkci $f(x,y)=\mathrm{e}^{x+y}$. $[x\in\mathbb{R};\ y\in\mathbb{R}]$ (viz poznámka).

$f(0;0)=\mathrm{e}^{0+0}=\mathrm{e}^0=1$

$f_x(0;0)=1 \quad f_x=\mathrm{e}^{x+y}\cdot 1=\mathrm{e}^{x+y}$
$f_y(0;0)=1 \quad f_y=\mathrm{e}^{x+y}\cdot 1=\mathrm{e}^{x+y}$

$\mathrm{d}f(0;0)=1\cdot(x-0)+1\cdot(y-0)=x+y$

$f_{xx}(0;0)=1 \quad f_{xx}=\mathrm{e}^{x+y}\cdot 1=\mathrm{e}^{x+y}$
$f_{xy}(0;0)=1 \quad f_{xy}=\mathrm{e}^{x+y}\cdot 1=\mathrm{e}^{x+y}$
$f_{yx}(0;0)=1 \quad f_{yx}=\mathrm{e}^{x+y}\cdot 1=\mathrm{e}^{x+y}$
$f_{yy}(0;0)=1 \quad f_{yy}=\mathrm{e}^{x+y}\cdot 1=\mathrm{e}^{x+y}$

$\mathrm{d}^2f(0;0)=1\cdot x\cdot x+1\cdot x\cdot y+1\cdot y\cdot x+1\cdot y\cdot y=x^2+2xy+y^2=(x+y)^2$

$f_{xxx}(0;0)=1 \quad f_{xxx}=\mathrm{e}^{x+y}$
$f_{xxy}(0;0)=1 \quad f_{xxy}=\mathrm{e}^{x+y}$
$f_{xyx}(0;0)=1 \quad f_{xyx}=\mathrm{e}^{x+y}$
$f_{xyy}(0;0)=1 \quad f_{xyy}=\mathrm{e}^{x+y}$
$f_{yxx}(0;0)=1 \quad f_{yxx}=\mathrm{e}^{x+y}$
$f_{yxy}(0;0)=1 \quad f_{yxy}=\mathrm{e}^{x+y}$
$f_{yyx}(0;0)=1 \quad f_{yyx}=\mathrm{e}^{x+y}$
$f_{yyy}(0;0)=1 \quad f_{yyy}=\mathrm{e}^{x+y}$

$$\mathrm{d}^3f(0;0)=1\cdot x\cdot x\cdot x+1\cdot x\cdot x\cdot y+1\cdot x\cdot y\cdot x+1\cdot x\cdot y\cdot y+1\cdot y\cdot x\cdot x+1\cdot y\cdot x\cdot y+$$
$$+1\cdot y\cdot y\cdot x+1\cdot y\cdot y\cdot y=x^3+3x^2y+3xy^2+y^3=(x+y)^3$$

$f_{xxxx}(0;0)=1 \quad f_{xxxx}=\mathrm{e}^{x+y}$
$f_{xxxy}(0;0)=1 \quad f_{xxxy}=\mathrm{e}^{x+y}$
$f_{xxyx}(0;0)=1 \quad f_{xxyx}=\mathrm{e}^{x+y}$
$f_{xxyy}(0;0)=1 \quad f_{xxyy}=\mathrm{e}^{x+y}$
$f_{xyxx}(0;0)=1 \quad f_{xyxx}=\mathrm{e}^{x+y}$
$f_{xyxy}(0;0)=1 \quad f_{xyxy}=\mathrm{e}^{x+y}$
$f_{xyyx}(0;0)=1 \quad f_{xyyx}=\mathrm{e}^{x+y}$
$f_{xyyy}(0;0)=1 \quad f_{xyyy}=\mathrm{e}^{x+y}$
$f_{yxxx}(0;0)=1 \quad f_{yxxx}=\mathrm{e}^{x+y}$
$f_{yxxy}(0;0)=1 \quad f_{yxxy}=\mathrm{e}^{x+y}$
$f_{yxyx}(0;0)=1 \quad f_{yxyx}=\mathrm{e}^{x+y}$
$f_{yxyy}(0;0)=1 \quad f_{yxyy}=\mathrm{e}^{x+y}$
$f_{yyxx}(0;0)=1 \quad f_{yyxx}=\mathrm{e}^{x+y}$
$f_{yyxy}(0;0)=1 \quad f_{yyxy}=\mathrm{e}^{x+y}$
$f_{yyyx}(0;0)=1 \quad f_{yyyx}=\mathrm{e}^{x+y}$
$f_{yyyy}(0;0)=1 \quad f_{yyyy}=\mathrm{e}^{x+y}$

$$\mathrm{d}^4f(0;0)=1\cdot x\cdot x\cdot x\cdot x+1\cdot x\cdot x\cdot x\cdot y+1\cdot x\cdot x\cdot y\cdot x+1\cdot x\cdot x\cdot y\cdot y+$$
$$+1\cdot x\cdot y\cdot x\cdot x+1\cdot x\cdot y\cdot x\cdot y+1\cdot x\cdot y\cdot y\cdot x+1\cdot x\cdot y\cdot y\cdot y+1\cdot y\cdot x\cdot x\cdot x+$$
$$+1\cdot y\cdot x\cdot x\cdot y+1\cdot y\cdot x\cdot y\cdot x+1\cdot y\cdot x\cdot y\cdot y+1\cdot y\cdot y\cdot x\cdot x+1\cdot y\cdot y\cdot x\cdot y+$$
$$+1\cdot y\cdot y\cdot y\cdot x+1\cdot y\cdot y\cdot y\cdot y=x^4+4x^3y+6x^2y^2+4xy^3+y^4=(x+y)^4$$

$$M=T(O)=1+\frac{x+y}{1!}+\frac{(x+y)^2}{2!}+\frac{(x+y)^3}{3!}+\frac{(x+y)^4}{4!}+\frac{(x+y)^5}{5!}+\frac{(x+y)^6}{6!}+\cdots$$

4. V okolí středu $B = [2; 1]$ rozviňte do **Taylorovy** řady funkci

$$f(x, y) = y^3 + 3x^2 - 3y^2 - 3x^2y + 12xy - 11x - 9y + 9\,. \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$$f(2;1) = 1^3 + 3\cdot 2^2 - 3\cdot 1^2 - 3\cdot 2^2\cdot 1 + 12\cdot 2\cdot 1 - 11\cdot 2 - 9\cdot 1 + 9 =$$
$$= 1 + 12 - 3 - 12 + 24 - 22 - 9 + 9 = 0$$

$f_x(2;1) = 1 \quad f_x = 6x - 6xy + 12y - 11$
$f_y(2;1) = 0 \quad f_y = 3y^2 - 6y - 3x^2 + 12x - 9$

$$\mathrm{d}f(2;1) = 1\cdot(x-2) + 0\cdot(y-1) = (x-2)$$

$f_{xx}(2;1) = 0 \quad f_{xx} = 6 - 6y$
$f_{xy}(2;1) = 0 \quad f_{xy} = -6x + 12$
$f_{yx}(2;1) = 0 \quad f_{yx} = -6x + 12$
$f_{yy}(2;1) = 0 \quad f_{yy} = 6y - 6$

$$\mathrm{d}^2 f(1;2) = 0\cdot(x-2)\cdot(x-2) + 0\cdot(x-2)\cdot(y-1) + 0\cdot(y-1)\cdot(x-2) +$$
$$+ 0\cdot(y-1)\cdot(y-1) = 0$$

$f_{xxx}(2;1) = 0 \quad f_{xxx} = 0$
$f_{xxy}(2;1) = -6 \quad f_{xxy} = -6$
$f_{xyx}(2;1) = -6 \quad f_{xyx} = -6$
$f_{xyy}(2;1) = 0 \quad f_{xyy} = 0$
$f_{yxx}(2;1) = -6 \quad f_{yxx} = -6$
$f_{yxy}(2;1) = 0 \quad f_{yxy} = 0$
$f_{yyx}(2;1) = 0 \quad f_{yyx} = 0$
$f_{yyy}(2;1) = 6 \quad f_{yyy} = 6$

$$\mathrm{d}^3 f(2;1) = 0 - 6\cdot(x-2)\cdot(x-2)\cdot(y-1) - 6\cdot(x-2)\cdot(y-1)\cdot(x-2) + 0 -$$
$$- 6\cdot(y-1)\cdot(x-2)\cdot(x-2) + 0 + 0 + 6\cdot(y-1)\cdot(y-1)\cdot(y-1) =$$
$$= 6(x-2)^3 - 18(x-2)(y-1)^2$$

Všechny další parciální derivace jsou rovny nule,

$$\mathrm{d}^4 f(2;1) = 0;\ \mathrm{d}^5 f(2;1) = 0;\ \mathrm{d}^6 f(2;1) = 0;\ \mathrm{d}^7 f(2;1) = 0;\ \ldots \quad \text{proto}$$

$$\underline{\underline{T(B) =}} \quad f(2;1) + \frac{\mathrm{d}f(2;1)}{1!} + \frac{\mathrm{d}^2 f(2;1)}{2!} + \frac{\mathrm{d}^3 f(2;1)}{3!} =$$
$$\underline{\underline{(x-2) - 3.(y-1).(x-2)^2 + (y-1)^3}}$$

5. Rozviňte do **Maclaurinovy** řady funkci $f(x,y) = \sin(x+y)$. $[x \in \mathbb{R};\ y \in \mathbb{R}]$ (viz poznámka).

$$f(0;0) = \sin(0+0) = \sin 0 = 0$$

$$f_x(0;0) = 1 \quad f_x = \cos(x+y)\cdot 1 = \cos(x+y)$$
$$f_y(0;0) = 1 \quad f_y = \cos(x+y)\cdot 1 = \cos(x+y)$$

$$\mathrm{d}f(0;0) = 1\cdot(x-0) + 1\cdot(y-0) = x+y$$

$$f_{xx}(0;0) = 0 \quad f_{xx} = -\sin(x+y)\cdot 1 = -\sin(x+y)$$
$$f_{xy}(0;0) = 0 \quad f_{xy} = -\sin(x+y)\cdot 1 = -\sin(x+y)$$
$$f_{yx}(0;0) = 0 \quad f_{yx} = -\sin(x+y)\cdot 1 = -\sin(x+y)$$
$$f_{yy}(0;0) = 0 \quad f_{yy} = -\sin(x+y)\cdot 1 = -\sin(x+y)$$

$$\mathrm{d}^2 f(0;0) = 0\cdot x\cdot x + 0\cdot x\cdot y + 0\cdot y\cdot x + 0\cdot y\cdot y = 0$$

$$f_{xxx}(0;0) = -1 \quad f_{xxx} = -\cos(x+y)$$
$$f_{xxy}(0;0) = -1 \quad f_{xxy} = -\cos(x+y)$$
$$f_{xyx}(0;0) = -1 \quad f_{xyx} = -\cos(x+y)$$
$$f_{xyy}(0;0) = -1 \quad f_{xyy} = -\cos(x+y)$$
$$f_{yxx}(0;0) = -1 \quad f_{yxx} = -\cos(x+y)$$
$$f_{yxy}(0;0) = -1 \quad f_{yxy} = -\cos(x+y)$$
$$f_{yyx}(0;0) = -1 \quad f_{yyx} = -\cos(x+y)$$
$$f_{yyy}(0;0) = -1 \quad f_{yyy} = -\cos(x+y)$$

$$\mathrm{d}^3 f(0;0) = -1\cdot x\cdot x\cdot x - 1\cdot x\cdot x\cdot y - 1\cdot x\cdot y\cdot x - 1\cdot x\cdot y\cdot y - 1\cdot y\cdot x\cdot x - 1\cdot y\cdot x\cdot y +$$
$$-1\cdot y\cdot y\cdot x - 1\cdot y\cdot y\cdot y = -x^3 - 3x^2y - 3xy^2 - y^3 = -(x+y)^3$$

$$f_{xxxx}(0;0) = 0 \quad f_{xxxx} = -[-\sin(x+y)] = \sin(x+y)$$
$$f_{xxxy}(0;0) = 0 \quad f_{xxxy} = \sin(x+y)$$
$$f_{xxyx}(0;0) = 0 \quad f_{xxyx} = \sin(x+y)$$
$$f_{xxyy}(0;0) = 0 \quad f_{xxyy} = \sin(x+y)$$
$$f_{xyxx}(0;0) = 0 \quad f_{xyxx} = \sin(x+y)$$
$$f_{xyxy}(0;0) = 0 \quad f_{xyxy} = \sin(x+y)$$
$$f_{xyyx}(0;0) = 0 \quad f_{xyyx} = \sin(x+y)$$
$$f_{xyyy}(0;0) = 0 \quad f_{xyyy} = \sin(x+y)$$
$$f_{yxxx}(0;0) = 0 \quad f_{yxxx} = \sin(x+y)$$
$$f_{yxxy}(0;0) = 0 \quad f_{yxxy} = \sin(x+y)$$
$$f_{yxyx}(0;0) = 0 \quad f_{yxyx} = \sin(x+y)$$
$$f_{yxyy}(0;0) = 0 \quad f_{yxyy} = \sin(x+y)$$
$$f_{yyxx}(0;0) = 0 \quad f_{yyxx} = \sin(x+y)$$
$$f_{yyxy}(0;0) = 0 \quad f_{yyxy} = \sin(x+y)$$
$$f_{yyyx}(0;0) = 0 \quad f_{yyyx} = \sin(x+y)$$
$$f_{yyyy}(0;0) = 0 \quad f_{yyyy} = \sin(x+y) \qquad f_{yyyyy} = \cos(x+y) \quad f_{yyyyy}(0;0) = 1$$

$$\mathrm{d}^4 f(0;0) = 0\cdot x\cdot x\cdot x\cdot x + 0\cdot x\cdot x\cdot x\cdot y + 0\cdot x\cdot x\cdot y\cdot x + 0\cdot x\cdot x\cdot y\cdot y +$$
$$+0\cdot x\cdot y\cdot x\cdot x + 0\cdot x\cdot y\cdot x\cdot y + 0\cdot x\cdot y\cdot y\cdot x + 0\cdot x\cdot y\cdot y\cdot y + 0\cdot y\cdot x\cdot x\cdot x +$$
$$+0\cdot y\cdot x\cdot x\cdot y + 0\cdot y\cdot x\cdot y\cdot x + 0\cdot y\cdot x\cdot y\cdot y + 0\cdot y\cdot y\cdot x\cdot x + 0\cdot y\cdot y\cdot x\cdot y +$$
$$+0\cdot y\cdot y\cdot y\cdot x + 0\cdot y\cdot y\cdot y\cdot y = 0$$

$$M = \frac{x+y}{1!} - \frac{(x+y)^3}{3!} + \frac{(x+y)^5}{5!} - \frac{(x+y)^7}{7!} + \cdots$$

Funkce sin je LICHÁ.

6. V okolí středu $A = [1; 2]$ rozviňte do **Taylorovy** řady funkci

$$f(x, y) = x^3 - 3x^2 + 3y^2 - 3xy^2 + 12xy - 9x - 11y + 9 . \qquad [x \in \mathbb{R}; \; y \in \mathbb{R}]$$

$$f(1; 2) = 1^3 - 3 \cdot 1^2 + 3 \cdot 2^2 - 3 \cdot 1 \cdot 2^2 + 12 \cdot 1 \cdot 2 - 9 \cdot 1 - 11 \cdot 2 + 9 =$$
$$= 1 - 3 + 12 - 12 + 24 - 9 - 22 + 9 = 0$$

$f_x(1; 2) = 0 \quad f_x = 3x^2 - 6x - 3y^2 + 12y - 9$
$f_y(1; 2) = 1 \quad f_y = 6y - 6xy + 12x - 11$

$$\mathrm{d}f(1; 2) = 0 \cdot (x - 1) + 1 \cdot (y - 2) = (y - 2)$$

$f_{xx}(1; 2) = 0 \quad f_{xx} = 6x - 6$
$f_{xy}(1; 2) = 0 \quad f_{xy} = -6y + 12$
$f_{yx}(1; 2) = 0 \quad f_{yx} = -6y + 12$
$f_{yy}(1; 2) = 0 \quad f_{yy} = 6 - 6x$

$$\mathrm{d}^2f(1; 2) = 0 \cdot (x - 1) \cdot (x - 1) + 0 \cdot (x - 1) \cdot (y - 2) + 0 \cdot (y - 2) \cdot (x - 1) +$$
$$+ 0 \cdot (y - 2) \cdot (y - 2) = 0$$

$f_{xxx}(1; 2) = 6 \quad f_{xxx} = 6$
$f_{xxy}(1; 2) = 0 \quad f_{xxy} = 0$
$f_{xyx}(1; 2) = 0 \quad f_{xyx} = 0$
$f_{xyy}(1; 2) = -6 \quad f_{xyy} = -6$
$f_{yxx}(1; 2) = 0 \quad f_{yxx} = 0$
$f_{yxy}(1; 2) = -6 \quad f_{yxy} = -6$
$f_{yyx}(1; 2) = -6 \quad f_{yyx} = -6$
$f_{yyy}(1; 2) = 0 \quad f_{yyy} = 0$

$$\mathrm{d}^3f(1; 2) = 6 \cdot (x - 1) \cdot (x - 1) \cdot (x - 1) + 0 + 0 - 6 \cdot (x - 1) \cdot (y - 2) \cdot (y - 2) + 0 -$$
$$- 6 \cdot (y - 2) \cdot (x - 1) \cdot (y - 2) - 6 \cdot (y - 2) \cdot (y - 2) \cdot (x - 1) + 0 =$$
$$= 6(x - 1)^3 - 18(x - 1)(y - 2)^2$$

Všechny další parciální derivace jsou rovny nule,

$$\mathrm{d}^6f(1; 2) = 0; \; \mathrm{d}^5f(1; 2) = 0; \; \mathrm{d}^6f(1; 2) = 0; \; \mathrm{d}^7f(1; 2) = 0; \; \dots \quad \text{proto}$$

$$\underline{\underline{T(A) =}} \quad f(1; 2) + \frac{\mathrm{d}f(1; 2)}{1!} + \frac{\mathrm{d}^2f(1; 2)}{2!} + \frac{\mathrm{d}^3f(1; 2)}{3!} =$$
$$\underline{\underline{(y - 2) + (x - 1)^3 - 3.(x - 1).(y - 2)^2}}$$

7. Rozviňte do **Maclaurinovy** řady funkci $f(x,y) = \mathrm{e}^{xy}$. $[x \in \mathbb{R};\ y \in \mathbb{R}]$

(viz poznámka).

$f(0;0) = \mathrm{e}^{0\cdot 0} = \mathrm{e}^0 = 1$

$f_x(0;0) = 0 \quad f_x = \mathrm{e}^{xy} \cdot y = y \cdot \mathrm{e}^{xy}$
$f_y(0;0) = 0 \quad f_y = \mathrm{e}^{xy} \cdot x = x \cdot \mathrm{e}^{xy}$

$\mathrm{d}f(0;0) = 0 \cdot (x-0) + 0 \cdot (y-0) = 0$

$f_{xx}(0;0) = 0 \quad f_{xx} = y \cdot \mathrm{e}^{xy} \cdot y = y^2 \cdot \mathrm{e}^{xy}$
$f_{xy}(0;0) = 1 \quad f_{xy} = 1 \cdot \mathrm{e}^{xy} + y \cdot \mathrm{e}^{xy} \cdot x = \mathrm{e}^{xy} + xy \cdot \mathrm{e}^{xy}$
$f_{yx}(0;0) = 1 \quad f_{yx} = 1 \cdot \mathrm{e}^{xy} + x \cdot \mathrm{e}^{xy} \cdot y = \mathrm{e}^{xy} + xy \cdot \mathrm{e}^{xy}$
$f_{yy}(0;0) = 0 \quad f_{yy} = x \cdot \mathrm{e}^{xy} \cdot x = x^2 \cdot \mathrm{e}^{xy}$

Je lepší výsledky před dalším derivováním co nejvíce upravit tak, jako v následujících příkladech v tomto dokumentu, nebo ve 3. příkladu, 9. cvičení na parciální derivace.

$\mathrm{d}^2 f(0;0) = 0 \cdot x \cdot x + 1 \cdot x \cdot y + 1 \cdot y \cdot x + 0 \cdot y \cdot y = 2xy = 2! \cdot \frac{xy}{1!}$

$f_{xxx}(0;0) = 0 \quad f_{xxx} = y^2 \cdot \mathrm{e}^{xy} \cdot y = y^3 \cdot \mathrm{e}^{xy}$
$f_{xxy}(0;0) = 0 \quad f_{xxy} = 2y \cdot \mathrm{e}^{xy} + y^2 \cdot \mathrm{e}^{xy} \cdot x = 2y \cdot \mathrm{e}^{xy} + xy^2 \cdot \mathrm{e}^{xy}$
$f_{xyx}(0;0) = 0 \quad f_{xyx} = \mathrm{e}^{xy} \cdot y + y \cdot \mathrm{e}^{xy} + xy \cdot \mathrm{e}^{xy} \cdot y = 2y \cdot \mathrm{e}^{xy} + xy^2 \cdot \mathrm{e}^{xy}$
$f_{xyy}(0;0) = 0 \quad f_{xyy} = \mathrm{e}^{xy} \cdot x + x \cdot \mathrm{e}^{xy} + xy \cdot \mathrm{e}^{xy} \cdot x = 2x \cdot \mathrm{e}^{xy} + x^2 y \cdot \mathrm{e}^{xy}$
$f_{yxx}(0;0) = 0 \quad f_{yxx} = \mathrm{e}^{xy} \cdot y + y \cdot \mathrm{e}^{xy} + xy \cdot \mathrm{e}^{xy} \cdot y = 2y \cdot \mathrm{e}^{xy} + xy^2 \cdot \mathrm{e}^{xy}$
$f_{yxy}(0;0) = 0 \quad f_{yxy} = \mathrm{e}^{xy} \cdot x + x \cdot \mathrm{e}^{xy} + xy \cdot \mathrm{e}^{xy} \cdot x = 2x \cdot \mathrm{e}^{xy} + x^2 y \cdot \mathrm{e}^{xy}$
$f_{yyx}(0;0) = 0 \quad f_{yyx} = 2x \cdot \mathrm{e}^{xy} + x^2 \cdot \mathrm{e}^{xy} \cdot y = 2x \cdot \mathrm{e}^{xy} + x^2 y \cdot \mathrm{e}^{xy}$
$f_{yyy}(0;0) = 0 \quad f_{yyy} = x^2 \cdot \mathrm{e}^{xy} \cdot x = x^3 \cdot \mathrm{e}^{xy}$

$\mathrm{d}^3 f(0;0) = 0 \cdot x \cdot x \cdot x + 0 \cdot x \cdot x \cdot y + 0 \cdot x \cdot y \cdot x + 0 \cdot x \cdot y \cdot y + 0 \cdot y \cdot x \cdot x + 0 \cdot y \cdot x \cdot y +$
$+ 0 \cdot y \cdot y \cdot x + 0 \cdot y \cdot y \cdot y = 0$

$f_{xxxx}(0;0) = 0 \quad f_{xxxx} = y^3 \cdot \mathrm{e}^{xy} \cdot y = y^4 \cdot \mathrm{e}^{xy}$
$f_{xxxy}(0;0) = 0 \quad f_{xxxy} = 3y^2 \cdot \mathrm{e}^{xy} + y^3 \cdot \mathrm{e}^{xy} \cdot x = 3y^2 \cdot \mathrm{e}^{xy} + xy^3 \cdot \mathrm{e}^{xy}$
$f_{xxyx}(0;0) = 0 \quad f_{xxyx} = 2y \cdot \mathrm{e}^{xy} \cdot y + y^2 \cdot \mathrm{e}^{xy} + xy^2 \cdot \mathrm{e}^{xy} \cdot y = 3y^2 \cdot \mathrm{e}^{xy} + xy^3 \cdot \mathrm{e}^{xy}$
$f_{xxyy}(0;0) = 2 \quad f_{xxyy} = 2 \cdot \mathrm{e}^{xy} + 2y \cdot \mathrm{e}^{xy} \cdot x + 2xy \cdot \mathrm{e}^{xy} + xy^2 \cdot \mathrm{e}^{xy} \cdot x =$
$= 2 \cdot \mathrm{e}^{xy} + 4xy \cdot \mathrm{e}^{xy} + x^2 y^2 \cdot \mathrm{e}^{xy}$
$f_{xyxx}(0;0) = 0 \quad f_{xyxx} = 2y \cdot \mathrm{e}^{xy} \cdot y + y^2 \cdot \mathrm{e}^{xy} + xy^2 \cdot \mathrm{e}^{xy} \cdot y = 3y^2 \cdot \mathrm{e}^{xy} + xy^3 \cdot \mathrm{e}^{xy}$
$f_{xyxy}(0;0) = 2 \quad f_{xyxy} = 2 \cdot \mathrm{e}^{xy} + 2y \cdot \mathrm{e}^{xy} \cdot x + 2xy \cdot \mathrm{e}^{xy} + xy^2 \cdot \mathrm{e}^{xy} \cdot x =$
$= 2 \cdot \mathrm{e}^{xy} + 4xy \cdot \mathrm{e}^{xy} + x^2 y^2 \cdot \mathrm{e}^{xy}$
$f_{xyyx}(0;0) = 2 \quad f_{xyyx} = 2 \cdot \mathrm{e}^{xy} + 2x \cdot \mathrm{e}^{xy} \cdot y + 2xy \cdot \mathrm{e}^{xy} + x^2 y \cdot \mathrm{e}^{xy} \cdot y =$
$= 2 \cdot \mathrm{e}^{xy} + 4xy \cdot \mathrm{e}^{xy} + x^2 y^2 \cdot \mathrm{e}^{xy}$
$f_{xyyy}(0;0) = 0 \quad f_{xyyy} = 2x \cdot \mathrm{e}^{xy} \cdot x + x^2 \cdot \mathrm{e}^{xy} + x^2 y \cdot \mathrm{e}^{xy} \cdot x = 3x^2 \cdot \mathrm{e}^{xy} + x^3 y \cdot \mathrm{e}^{xy}$

$f_{yxxx}(0;0)=0 \quad f_{yxxx}=2y\cdot e^{xy}\cdot y+y^2\cdot e^{xy}+xy^2\cdot e^{xy}\cdot y=3y^2\cdot e^{xy}+xy^3\cdot e^{xy}$

$f_{yxxy}(0;0)=2 \quad f_{yxxy}=2\cdot e^{xy}+2y\cdot e^{xy}\cdot x+2xy\cdot e^{xy}+xy^2\cdot e^{xy}\cdot x=$
$=2\cdot e^{xy}+4xy\cdot e^{xy}+x^2y^2\cdot e^{xy}$

$f_{yxyx}(0;0)=2 \quad f_{yxyx}=2\cdot e^{xy}+2x\cdot e^{xy}\cdot y+2xye^{xy}+x^2y\cdot e^{xy}\cdot y=$
$=2\cdot e^{xy}+4xy\cdot e^{xy}+x^2y^2\cdot e^{xy}$

$f_{yxyy}(0;0)=0 \quad f_{yxyy}=2x\cdot e^{xy}\cdot x+x^2\cdot e^{xy}+x^2y\cdot e^{xy}\cdot x=3x^2\cdot e^{xy}+x^3y\cdot e^{xy}$

$f_{yyxx}(0;0)=2 \quad f_{yyxx}=2\cdot e^{xy}+2x\cdot e^{xy}\cdot y+2xy\cdot e^{xy}+x^2y\cdot e^{xy}\cdot y=$
$=2\cdot e^{xy}+4xy\cdot e^{xy}+x^2y^2\cdot e^{xy}$

$f_{yyxy}(0;0)=0 \quad f_{yyxy}=2x\cdot e^{xy}\cdot x+x^2\cdot e^{xy}+x^2y\cdot e^{xy}\cdot x=3x^2\cdot e^{xy}+x^3y\cdot e^{xy}$

$f_{yyyx}(0;0)=0 \quad f_{yyyx}=3x^2\cdot e^{xy}+x^3\cdot e^{xy}\cdot y=3x^2\cdot e^{xy}+x^3y\cdot e^{xy}$

$f_{yyyy}(0;0)=0 \quad f_{yyyy}=x^3\cdot e^{xy}\cdot x=x^4\cdot e^{xy}$

$$\begin{aligned}d^4f(0;0)&=0\cdot x\cdot x\cdot x\cdot x+0\cdot x\cdot x\cdot x\cdot y+0\cdot x\cdot x\cdot y\cdot x+2\cdot x\cdot x\cdot y\cdot y+\\&+0\cdot x\cdot y\cdot x\cdot x+2\cdot x\cdot y\cdot x\cdot y+2\cdot x\cdot y\cdot y\cdot x+0\cdot x\cdot y\cdot y\cdot y+0\cdot y\cdot x\cdot x\cdot x+\\&+2\cdot y\cdot x\cdot x\cdot y+2\cdot y\cdot x\cdot y\cdot x+0\cdot y\cdot x\cdot y\cdot y+2\cdot y\cdot y\cdot x\cdot x+0\cdot y\cdot y\cdot x\cdot y+\\&+0\cdot y\cdot y\cdot y\cdot x+0\cdot y\cdot y\cdot y\cdot y=6\cdot(2x^2y^2)=\\&=12x^2y^2=\underbrace{4\cdot 3}_{12}\cdot\frac{2!}{2!}\cdot x^2y^2=\frac{4!}{2!}\cdot x^2y^2=4!\cdot\frac{x^2y^2}{2!}\end{aligned}$$

$f_{xxxxx}(0;0)=0 \quad f_{xxxxx}=y^4\cdot e^{xy}\cdot y=y^5\cdot e^{xy}$

$\vdots$

$$d^5f(0;0)=0\cdot x\cdot x\cdot x\cdot x\cdot x+\ldots=0$$

$$\underline{\underline{M=}}\quad f(0;0)+\frac{df(0;0)}{1!}+\frac{d^2f(0;0)}{2!}+\frac{d^3f(0;0)}{3!}+\cdots=1+0+\frac{2!\cdot\frac{xy}{1!}}{2!}+0+\frac{4!\cdot\frac{x^2y^2}{2!}}{4!}+\ldots=$$

$$\underline{\underline{=1+\frac{xy}{1!}+\frac{(xy)^2}{2!}+\frac{(xy)^3}{3!}+\frac{(xy)^4}{4!}+\frac{(xy)^5}{5!}+\frac{(xy)^6}{6!}+\cdots}}$$

8. V okolí středu $C = [1; -1]$ rozviňte do **Taylorovy** řady funkci

$$f(x, y) = x^3 + xy^2 - 3x + 2xy + 1\,. \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$f(1; -1) = 1^3 + 1 \cdot (-1)^2 - 3 \cdot 1 + 2 \cdot 1 \cdot (-1) + 1 = -2$

$$\begin{aligned} f_x(1; -1) &= -1 & f_x &= 3x^2 + y^2 - 3 + 2y \\ f_y(1; -1) &= 0 & f_y &= 2xy + 2x \end{aligned}$$

$\mathrm{d}f(1; -1) = -1 \cdot (x - 1) + 0 \cdot (y + 1) = -(x - 1)$

$$\begin{aligned} f_{xx}(1; -1) &= 6 & f_{xx} &= 6x \\ f_{xy}(1; -1) &= 0 & f_{xy} &= 2y + 2 \\ f_{yx}(1; -1) &= 0 & f_{yx} &= 2y + 2 \\ f_{yy}(1; -1) &= 2 & f_{yy} &= 2x \end{aligned}$$

$$\mathrm{d}^2 f(1; -1) = 6 \cdot (x - 1) \cdot (x - 1) + 0 \cdot (x - 1) \cdot (y + 1) + 0 \cdot (y + 1) \cdot (x - 1) + \\ + 2 \cdot (y + 1) \cdot (y + 2) = 6(x - 1)^2 + 2(y + 1)^2$$

$$\begin{aligned} f_{xxx}(1; -1) &= 6 & f_{xxx} &= 6 \\ f_{xxy}(1; -1) &= 0 & f_{xxy} &= 0 \\ f_{xyx}(1; -1) &= 0 & f_{xyx} &= 0 \\ f_{xyy}(1; -1) &= 2 & f_{xyy} &= 2 \\ f_{yxx}(1; -1) &= 0 & f_{yxx} &= 0 \\ f_{yxy}(1; -1) &= 2 & f_{yxy} &= 2 \\ f_{yyx}(1; -1) &= 2 & f_{yyx} &= 2 \\ f_{yyy}(1; -1) &= 0 & f_{yyy} &= 0 \end{aligned}$$

$$\mathrm{d}^3 f(1; -1) = 6 \cdot (x - 1) \cdot (x - 1) \cdot (x - 1) + 0 + 0 + 2 \cdot (x - 1) \cdot (y + 1) \cdot (y + 1) + 0 - \\ + 2 \cdot (y + 1) \cdot (x - 1) \cdot (y + 1) + 2 \cdot (y + 1) \cdot (y + 1) \cdot (x - 1) + 0 = \\ = 6(x - 1)^3 + 6(x - 1)(y + 1)^2$$

Všechny další parciální derivace jsou rovny nule,

$$\mathrm{d}^4 f(1; -1) = 0;\ \mathrm{d}^5 f(1; -1) = 0;\ \mathrm{d}^6 f(1; -1) = 0;\ \mathrm{d}^7 f(1; -1) = 0;\ \ldots \quad \text{proto}$$

$$\underline{\underline{T(C) =}} \quad f(1; -1) + \frac{\mathrm{d}f(1; -1)}{1!} + \frac{\mathrm{d}^2 f(1; -1)}{2!} + \frac{\mathrm{d}^3 f(1; -1)}{3!} =$$

$$\underline{\underline{-2 - (x - 1) + 3(x - 1)^2 + (y + 1)^2 + (x - 1)^3 + (x - 1)(y + 1)^2}}$$

9. Rozviňte do **Maclaurinovy** řady funkci $f(x,y) = \cos(xy)$. $[x \in \mathbb{R};\ y \in \mathbb{R}]$

(viz poznámka).

$f(0;0) = \cos(0 \cdot 0) = \cos 0 = 1$

$f_x(0;0) = 0 \quad f_x = -\sin(xy) \cdot y = -y \cdot \sin(xy)$
$f_y(0;0) = 0 \quad f_y = -\sin(xy) \cdot x = -x \cdot \sin(xy)$

$\mathrm{d}f(0;0) = 0 \cdot (x-0) + 0 \cdot (y-0) = 0$

$f_{xx}(0;0) = 0 \quad f_{xx} = -y \cdot \cos(xy) \cdot y = -y^2 \cdot \cos(xy)$
$f_{xy}(0;0) = 0 \quad f_{xy} = -1 \cdot \sin(xy) - y \cdot \cos(xy) \cdot x = -\sin(xy) - xy \cdot \cos(xy)$
$f_{yx}(0;0) = 0 \quad f_{yx} = -1 \cdot \sin(xy) - x \cdot \cos(xy) \cdot y = -\sin(xy) - xy \cdot \cos(xy)$
$f_{yy}(0;0) = 0 \quad f_{yy} = -x \cdot \cos(xy) \cdot x = -x^2 \cdot \cos(xy)$

$\mathrm{d}^2 f(0;0) = 0 \cdot x \cdot x + 0 \cdot x \cdot y + 0 \cdot y \cdot x + 0 \cdot y \cdot y = 0$

$f_{xxx}(0;0) = 0 \quad f_{xxx} = -y^2 \cdot [-\sin(xy) \cdot y] = y^3 \cdot \sin(xy)$
$f_{xxy}(0;0) = 0 \quad f_{xxy} = -2y \cdot \cos(xy) - y^2 \cdot [-\sin(xy) \cdot x] = xy^2 \cdot \sin(xy) - 2y \cdot \cos(xy)$
$f_{xyx}(0;0) = 0 \quad f_{xyx} = -\cos(xy) \cdot y - y \cdot \cos(xy) - xy \cdot [-\sin(xy) \cdot y] = xy^2 \cdot \sin(xy) - 2y \cdot \cos(xy)$
$f_{xyy}(0;0) = 0 \quad f_{xyy} = -\cos(xy) \cdot x - x \cdot \cos(xy) - xy \cdot [-\sin(xy) \cdot x] = x^2 y \cdot \sin(xy) - 2x \cdot \cos(xy)$
$f_{yxx}(0;0) = 0 \quad f_{yxx} = -\cos(xy) \cdot y - y \cdot \cos(xy) - xy \cdot [-\sin(xy) \cdot y] = xy^2 \cdot \sin(xy) - 2y \cdot \cos(xy)$
$f_{yxy}(0;0) = 0 \quad f_{yxy} = -\cos(xy) \cdot x - x \cdot \cos(xy) - xy \cdot [-\sin(xy) \cdot x] = x^2 y \cdot \sin(xy) - 2x \cdot \cos(xy)$
$f_{yyx}(0;0) = 0 \quad f_{yyx} = -2x \cdot \cos(xy) - x^2 \cdot [-\sin(xy) \cdot y] = x^2 y \cdot \sin(xy) - 2x \cdot \cos(xy)$
$f_{yyy}(0;0) = 0 \quad f_{yyy} = -x^2 \cdot [-\sin(xy) \cdot x] = x^3 \cdot \sin(xy)$

$$\mathrm{d}^3 f(0;0) = 0 \cdot x \cdot x \cdot x + 0 \cdot x \cdot x \cdot y + 0 \cdot x \cdot y \cdot x + 0 \cdot x \cdot y \cdot y + 0 \cdot y \cdot x \cdot x + 0 \cdot y \cdot x \cdot y + 0 \cdot y \cdot y \cdot x + 0 \cdot y \cdot y \cdot y = 0$$

$f_{xxxx}(0;0) = 0 \quad f_{xxxx} = y^3 \cdot \cos(xy) \cdot y = y^4 \cdot \cos(xy)$
$f_{xxxy}(0;0) = 0 \quad f_{xxxy} = 3y^2 \cdot \sin(xy) + y^3 \cdot \cos(xy) \cdot x = 3y^2 \cdot \sin(xy) + xy^3 \cdot \cos(xy)$
$f_{xxyx}(0;0) = 0 \quad f_{xxyx} = y^2 \cdot \sin(xy) + xy^2 \cdot \cos(xy) \cdot y - 2y \cdot [-\sin(xy) \cdot y] = 3y^2 \cdot \sin(xy) + xy^3 \cdot \cos(xy)$
$f_{xxyy}(0;0) = -2 \quad f_{xxyy} = 2xy \cdot \sin(xy) + xy^2 \cdot \cos(xy) \cdot x - 2 \cdot \cos(xy) - 2y \cdot [-\sin(xy) \cdot x] = 4xy \cdot \sin(xy) + (x^2y^2 - 2) \cdot \cos(xy)$

$f_{xyxx}(0;0) = 0$ $\quad f_{xyxx} = y^2 \cdot \sin(xy) + xy^2 \cdot \cos(xy) \cdot y - 2y \cdot [-\sin(xy) \cdot y] = 3y^2 \cdot \sin(xy) + xy^3 \cdot \cos(xy)$

$f_{xyxy}(0;0) = -2$ $\quad f_{xyxy} = 2xy \cdot \sin(xy) + xy^2 \cos(xy) \cdot x - 2\cos(xy) - 2y \cdot [-\sin(xy) \cdot x] = 4xy \cdot \sin(xy) + (x^2y^2 - 2) \cdot \cos(xy)$

$f_{xyyx}(0;0) = -2$ $\quad f_{xyyx} = 2xy \cdot \sin(xy) + x^2y \cdot \cos(xy) \cdot y - 2 \cdot \cos(xy) - 2x \cdot [-\sin(xy) \cdot y] = 4xy \cdot \sin(xy) + (x^2y^2 - 2) \cdot \cos(xy)$

$f_{xyyy}(0;0) = 0$ $\quad f_{xyyy} = x^2 \cdot \sin(xy) + x^2y \cdot \cos(xy) \cdot x - 2x \cdot [-\sin(xy) \cdot x] = 3x^2 \cdot \sin(xy) + x^3y \cdot \cos(xy)$

$f_{yxxx}(0;0) = 0$ $\quad f_{yxxx} = y^2 \cdot \sin(xy) + xy^2 \cdot \cos(xy) \cdot y - 2y \cdot [-\sin(xy) \cdot y] = 3y^2 \cdot \sin(xy) + xy^3 \cdot \cos(xy)$

$f_{yxxy}(0;0) = -2$ $\quad f_{yxxy} = 2xy \cdot \sin(xy) + xy^2 \cdot \cos(xy) \cdot x - 2 \cdot \cos(xy) - 2y \cdot [-\sin(xy) \cdot x] = 4xy \cdot \sin(xy) + (x^2y^2 - 2) \cdot \cos(xy)$

$f_{yxyx}(0;0) = -2$ $\quad f_{yxyx} = 2xy \cdot \sin(xy) + x^2y \cdot \cos(xy) \cdot y - 2 \cdot \cos(xy) - 2x \cdot [-\sin(xy) \cdot y] = 4xy \cdot \sin(xy) + (x^2y^2 - 2) \cdot \cos(xy)$

$f_{yxyy}(0;0) = 0$ $\quad f_{yxyy} = x^2 \cdot \sin(xy) + x^2y \cdot \cos(xy) \cdot x - 2x \cdot [-\sin(xy) \cdot x] = 3x^2 \cdot \sin(xy) + x^3y \cdot \cos(xy)$

$f_{yyxx}(0;0) = -2$ $\quad f_{yyxx} = 2xy \cdot \sin(xy) + x^2y \cdot \cos(xy) \cdot y - 2 \cdot \cos(xy) - 2x \cdot [-\sin(xy) \cdot y] = 4xy \cdot \sin(xy) + (x^2y^2 - 2) \cdot \cos(xy)$

$f_{yyxy}(0;0) = 0$ $\quad f_{yyxy} = x^2 \cdot \sin(xy) + x^2y \cdot \cos(xy) \cdot x - 2x \cdot [-\sin(xy) \cdot x] = 3x^2 \cdot \sin(xy) + x^3y \cdot \cos(xy)$

$f_{yyyx}(0;0) = 0$ $\quad f_{yyyx} = 3x^2 \cdot \sin(xy) + x^3 \cdot \cos(xy) \cdot y = 3x^2 \cdot \sin(xy) + x^3y \cdot \cos(xy)$

$f_{yyyy}(0;0) = 0$ $\quad f_{yyyy} = x^3 \cdot \cos(xy) \cdot x = x^4 \cdot \cos(xy)$

$$\begin{aligned}
\mathrm{d}^4 f(0;0) &= 0 \cdot x \cdot x \cdot x \cdot x + 0 \cdot x \cdot x \cdot x \cdot y + 0 \cdot x \cdot x \cdot y \cdot x - 2 \cdot x \cdot x \cdot y \cdot y + \\
&+ 0 \cdot x \cdot y \cdot x \cdot x - 2 \cdot x \cdot y \cdot x \cdot y - 2 \cdot x \cdot y \cdot y \cdot x + 0 \cdot x \cdot y \cdot y \cdot y + 0 \cdot y \cdot x \cdot x \cdot x + \\
&- 2 \cdot y \cdot x \cdot x \cdot y - 2 \cdot y \cdot x \cdot y \cdot x + 0 \cdot y \cdot x \cdot y \cdot y - 2 \cdot y \cdot y \cdot x \cdot x + 0 \cdot y \cdot y \cdot x \cdot y + \\
&+ 0 \cdot y \cdot y \cdot y \cdot x + 0 \cdot y \cdot y \cdot y \cdot y = 6 \cdot (-2x^2y^2) = \\
&= -12x^2y^2 = -\underbrace{4 \cdot 3}_{12} \cdot \frac{2!}{2!} \cdot x^2y^2 = -\frac{4!}{2!} \cdot x^2y^2 = -4! \cdot \frac{x^2y^2}{2!}
\end{aligned}$$

$f_{xxxxx}(0;0) = 0$ $\quad f_{xxxxx} = y^4 \cdot \cos(xy) \cdot y = y^5 \cdot \cos(xy)$

$\vdots$

$$\mathrm{d}^5 f(0;0) = 0 \cdot x \cdot x \cdot x \cdot x \cdot x + \ldots = 0$$

$$\underline{\underline{M =}} \quad f(0;0) + \frac{\mathrm{d}f(0;0)}{1!} + \frac{\mathrm{d}^2 f(0;0)}{2!} + \frac{\mathrm{d}^3 f(0;0)}{3!} + \cdots = 1 + 0 + 0 + 0 + \frac{-4! \cdot \frac{x^2y^2}{2!}}{4!} + 0 + \ldots =$$

$$\underline{\underline{= 1 - \frac{(xy)^2}{2!} + \frac{(xy)^4}{4!} - \frac{(xy)^6}{6!} + \cdots}}$$

Funkce cos je SUDÁ.

10. V okolí středu $D = [-1; 1]$ rozviňte do **Taylorovy** řady funkci

$$f(x,y) = y^3 + x^2y + 2xy - 3y + 1\,. \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$f(-1;1) = 1^3 + (-1)^2 \cdot 1 + 2 \cdot (-1) \cdot 1 - 3 \cdot 1 + 1 = -2$

$f_x(-1;1) = 0 \quad f_x = 2xy + 2y$
$f_y(-1;1) = -1 \quad f_y = 3y^2 + x^2 + 2x - 3$

$\mathrm{d}f(-1;1) = 0 \cdot (x+1) - 1 \cdot (y-1) = -(y-1)$

$f_{xx}(-1;1) = 2 \quad f_{xx} = 2y$
$f_{xy}(-1;1) = 0 \quad f_{xy} = 2x + 2$
$f_{yx}(-1;1) = 0 \quad f_{yx} = 2x + 2$
$f_{yy}(-1;1) = 6 \quad f_{yy} = 6y$

$$\mathrm{d}^2f(-1;1) = 2 \cdot (x+1) \cdot (x+1) + 0 \cdot (x+1) \cdot (y-1) + 0 \cdot (y-1) \cdot (x+1) + \\ + 6 \cdot (y-1) \cdot (y-1) = 2(x+1)^2 + 6(y-1)^2$$

$f_{xxx}(-1;1) = 0 \quad f_{xxx} = 0$
$f_{xxy}(-1;1) = 2 \quad f_{xxy} = 2$
$f_{xyx}(-1;1) = 2 \quad f_{xyx} = 2$
$f_{xyy}(-1;1) = 0 \quad f_{xyy} = 0$
$f_{yxx}(-1;1) = 2 \quad f_{yxx} = 2$
$f_{yxy}(-1;1) = 0 \quad f_{yxy} = 0$
$f_{yyx}(-1;1) = 0 \quad f_{yyx} = 0$
$f_{yyy}(-1;1) = 6 \quad f_{yyy} = 6$

$$\mathrm{d}^3f(-1;1) = 0 \cdot (x+1) \cdot (x+1) \cdot (x+1) + 2 \cdot (x+1) \cdot (x+1) \cdot (y-1) \\ + 2 \cdot (x+1) \cdot (y-1) \cdot (x+1) + 0 + 2 \cdot (y-1) \cdot (x+1) \cdot (x+1) + 0 + 0 + \\ + 6 \cdot (y-1) \cdot (y-1) \cdot (y-1) = 6(x+1)^2(y-1) + 6(y-1)^3$$

Všechny další parciální derivace jsou rovny nule,

$\mathrm{d}^4f(-1;1) = 0;\ \mathrm{d}^5f(-1;1) = 0;\ \mathrm{d}^6f(-1;1) = 0;\ \mathrm{d}^7f(-1;1) = 0;\ \ldots$ proto

$$\underline{\underline{T(D) =}} \quad f(-1;1) + \frac{\mathrm{d}f(-1;1)}{1!} + \frac{\mathrm{d}^2f(-1;1)}{2!} + \frac{\mathrm{d}^3f(-1;1)}{3!} =$$

$$\underline{\underline{-2 - (y-1) + (x+1)^2 + 3(y-1)^2 + (x+1)^2(y-1) + (y-1)^3}}$$

11. Rozviňte do **Maclaurinovy** řady funkci $f(x,y) = \mathrm{e}^{x^2+y^2}$. $[x \in \mathbb{R};\ y \in \mathbb{R}]$

(viz poznámka).

$f(0;0) = \mathrm{e}^{0^2+0^2} = \mathrm{e}^0 = 1$

$f_x(0;0) = 0 \quad f_x = \mathrm{e}^{x^2+y^2} \cdot 2x = 2x \cdot \mathrm{e}^{x^2+y^2}$

$f_y(0;0) = 0 \quad f_y = \mathrm{e}^{x^2+y^2} \cdot 2y = 2y \cdot \mathrm{e}^{x^2+y^2}$

$\mathrm{d}f(0;0) = 0 \cdot (x-0) + 0 \cdot (y-0) = 0$

$f_{xx}(0;0) = 2 \quad f_{xx} = 2\mathrm{e}^{x^2+y^2} + 2x\mathrm{e}^{x^2+y^2} \cdot 2x = (2+4x^2) \cdot \mathrm{e}^{x^2+y^2}$

$f_{xy}(0;0) = 0 \quad f_{xy} = 2x\mathrm{e}^{x^2+y^2} \cdot 2y = 4xy \cdot \mathrm{e}^{x^2+y^2}$

$f_{yx}(0;0) = 0 \quad f_{yx} = 2y\mathrm{e}^{x^2+y^2} \cdot 2x = 4xy \cdot \mathrm{e}^{x^2+y^2}$

$f_{yy}(0;0) = 2 \quad f_{yy} = 2\mathrm{e}^{x^2+y^2} + 2y\mathrm{e}^{x^2+y^2} \cdot 2y = (2+4y^2) \cdot \mathrm{e}^{x^2+y^2}$

$$\mathrm{d}^2 f(0;0) = 2x^2 + 2y^2 = 2 \cdot (x^2+y^2) = 2 \cdot \frac{1!}{1!} \cdot (x^2+y^2) = \frac{2!}{1!} \cdot (x^2+y^2) = 2! \cdot \frac{x^2+y^2}{1!}$$

$f_{xxx}(0;0) = 0 \quad f_{xxx} = 8x\mathrm{e}^{x^2+y^2} + (2+4x^2)\mathrm{e}^{x^2+y^2} \cdot 2x = (12x + 8x^3) \cdot \mathrm{e}^{x^2+y^2}$

$f_{xxy}(0;0) = 0 \quad f_{xxy} = (2+4x^2)\mathrm{e}^{x^2+y^2} \cdot 2y = (4y + 8x^2 y) \cdot \mathrm{e}^{x^2+y^2}$

$f_{xyx}(0;0) = 0 \quad f_{xyx} = 4y\mathrm{e}^{x^2+y^2} + 4xy\mathrm{e}^{x^2+y^2} \cdot 2x = (4y + 8x^2 y) \cdot \mathrm{e}^{x^2+y^2}$

$f_{xyy}(0;0) = 0 \quad f_{xyy} = 4x\mathrm{e}^{x^2+y^2} + 4xy\mathrm{e}^{x^2+y^2} \cdot 2y = (4x + 8xy^2) \cdot \mathrm{e}^{x^2+y^2}$

$f_{yxx}(0;0) = 0 \quad f_{yxx} = 4y\mathrm{e}^{x^2+y^2} + 4xy\mathrm{e}^{x^2+y^2} \cdot 2x = (4y + 8x^2 y) \cdot \mathrm{e}^{x^2+y^2}$

$f_{yxy}(0;0) = 0 \quad f_{yxy} = 4x\mathrm{e}^{x^2+y^2} + 4xy\mathrm{e}^{x^2+y^2} \cdot 2y = (4x + 8xy^2) \cdot \mathrm{e}^{x^2+y^2}$

$f_{yyx}(0;0) = 0 \quad f_{yyx} = (2+4y^2)\mathrm{e}^{x^2+y^2} \cdot 2x = (4x + 8xy^2) \cdot \mathrm{e}^{x^2+y^2}$

$f_{yyy}(0;0) = 0 \quad f_{yyy} = 8y\mathrm{e}^{x^2+y^2} + (2+4y^2)\mathrm{e}^{x^2+y^2} \cdot 2y = (12y + 8y^3) \cdot \mathrm{e}^{x^2+y^2}$

$\mathrm{d}^3 f(0;0) = 0$

$f_{xxxx}(0;0) = 12 \quad f_{xxxx} = (12 + 24x^2)\mathrm{e}^{x^2+y^2} + (12x + 8x^3)\mathrm{e}^{x^2+y^2} \cdot 2x =$

$= (12 + 48x^2 + 16x^4) \cdot \mathrm{e}^{x^2+y^2}$

$f_{xxxy}(0;0) = 0 \quad f_{xxxy} = (12x + 8x^3)\mathrm{e}^{x^2+y^2} \cdot 2y = (24xy + 16x^3 y) \cdot \mathrm{e}^{x^2+y^2}$

$f_{xxyx}(0;0) = 0 \quad f_{xxyx} = 16xy\mathrm{e}^{x^2+y^2} + (4y + 8x^2 y)\mathrm{e}^{x^2+y^2} \cdot 2x =$

$= (24xy + 16x^3 y) \cdot \mathrm{e}^{x^2+y^2}$

$f_{xxyy}(0;0) = 4 \quad f_{xxyy} = (4 + 8x^2)\mathrm{e}^{x^2+y^2} + (4y + 8x^2 y)\mathrm{e}^{x^2+y^2} \cdot 2y =$

$= (4 + 8x^2 + 8y^2 + 16x^2 y^2) \cdot \mathrm{e}^{x^2+y^2}$

$f_{xyxx}(0;0) = 0 \quad f_{xyxx} = 16xy\mathrm{e}^{x^2+y^2} + (4y + 8x^2 y)\mathrm{e}^{x^2+y^2} \cdot 2x =$

$= (24xy + 16x^3 y) \cdot \mathrm{e}^{x^2+y^2}$

$f_{xyxy}(0;0) = 4 \quad f_{xyxy} = (4 + 8x^2)\mathrm{e}^{x^2+y^2} + (4y + 8x^2 y)\mathrm{e}^{x^2+y^2} \cdot 2y =$

$= (4 + 8x^2 + 8y^2 + 16x^2 y^2) \cdot \mathrm{e}^{x^2+y^2}$

$f_{xyyx}(0;0) = 4 \quad f_{xyyx} = (4 + 8y^2)\mathrm{e}^{x^2+y^2} + (4x + 8xy^2)\mathrm{e}^{x^2+y^2} \cdot 2x =$

$= (4 + 8x^2 + 8y^2 + 16x^2 y^2) \cdot \mathrm{e}^{x^2+y^2}$

$f_{xyyy}(0;0) = 0 \quad f_{xyyy} = 16xy\mathrm{e}^{x^2+y^2} + (4x + 8xy^2)\mathrm{e}^{x^2+y^2} \cdot 2y =$

$= (24xy + 16xy^3) \cdot \mathrm{e}^{x^2+y^2}$

$f_{yxxx}(0;0) = 0$ $\quad f_{yxxx} = 16xye^{x^2+y^2} + (4y + 8x^2y)e^{x^2+y^2} \cdot 2x = (24xy + 16x^3y) \cdot e^{x^2+y^2}$

$f_{yxxy}(0;0) = 4$ $\quad f_{yxxy} = (4 + 8x^2)e^{x^2+y^2} + (4y + 8x^2y)e^{x^2+y^2} \cdot 2y = (4 + 8x^2 + 8y^2 + 16x^2y^2) \cdot e^{x^2+y^2}$

$f_{yxyx}(0;0) = 4$ $\quad f_{yxyx} = (4 + 8y^2)e^{x^2+y^2} + (4x + 8xy^2)e^{x^2+y^2} \cdot 2x = (4 + 8x^2 + 8y^2 + 16x^2y^2) \cdot e^{x^2+y^2}$

$f_{yxyy}(0;0) = 0$ $\quad f_{yxyy} = 16xye^{x^2+y^2} + (4x + 8xy^2)e^{x^2+y^2} \cdot 2y = (24xy + 16xy^3) \cdot e^{x^2+y^2}$

$f_{yyxx}(0;0) = 4$ $\quad f_{yyxx} = (4 + 8y^2)e^{x^2+y^2} + (4x + 8xy^2)e^{x^2+y^2} \cdot 2x = (4 + 8x^2 + 8y^2 + 16x^2y^2) \cdot e^{x^2+y^2}$

$f_{yyxy}(0;0) = 0$ $\quad f_{yyxy} = 16xye^{x^2+y^2} + (4x + 8xy^2)e^{x^2+y^2} \cdot 2y = (24xy + 16xy^3) \cdot e^{x^2+y^2}$

$f_{yyyx}(0;0) = 0$ $\quad f_{yyyx} = (12y + 8y^3)e^{x^2+y^2} \cdot 2x = (24xy + 16xy^3) \cdot e^{x^2+y^2}$

$f_{yyyy}(0;0) = 12$ $\quad f_{yyyy} = (12 + 24y^2)e^{x^2+y^2} + (12y + 8y^3)e^{x^2+y^2} \cdot 2y = (12 + 48y^2 + 16y^4) \cdot e^{x^2+y^2}$

$$\begin{aligned}
d^4f(0;0) &= 12xxxx + 0 + 0 + 4xxyy + 0 + 4xyxy + 4xyyx + 0 + 0 + 4yxxy + 4yxyx + 0 + \\
&\quad + 4yyxx + 0 + 0 + 12yyyy = \\
&= 12(x^4 + 2x^2y^2 + y^4) = \underbrace{4 \cdot 3}_{12} \cdot \frac{2!}{2!} \cdot (x^2 + y^2)^2 = \frac{4!}{2!} \cdot (x^2 + y^2)^2
\end{aligned}$$

$$\begin{aligned}
\underline{\underline{M =}} \quad & f(0;0) + \frac{df(0;0)}{1!} + \frac{d^2f(0;0)}{2!} + \frac{d^3f(0;0)}{3!} + \frac{d^4f(0;0)}{4!} + \cdots = \\
&= 1 + 0 + \frac{2! \cdot \frac{(x^2+y^2)}{1!}}{2!} + 0 + \frac{4! \cdot \frac{(x^2+y^2)^2}{2!}}{4!} + 0 + \frac{6! \cdot \frac{(x^2+y^2)^3}{3!}}{6!} + \ldots = \\
&\underline{\underline{= 1 + \frac{(x^2 + y^2)}{1!} + \frac{(x^2 + y^2)^2}{2!} + \frac{(x^2 + y^2)^3}{3!} + \frac{(x^2 + y^2)^4}{4!} + \frac{(x^2 + y^2)^5}{5!} + \cdots}}
\end{aligned}$$

12. V okolí středu $E=[1;-3]$ rozviňte do **Taylorovy** řady funkci

$$f(x,y)=x^3+xy-y-1\,. \qquad [x\in\mathbb{R}\,;\ y\in\mathbb{R}]$$

$f(1;-3)=1^3+1\cdot(-3)-(-3)-1=0$

$f_x(1;-3)=0 \quad f_x=3x^2+y$
$f_y(1;-3)=0 \quad f_y=x-1$

$\mathrm{d}f(1;-3)=0\cdot(x-1)+0\cdot(y+3)=0$

$f_{xx}(1;-3)=6 \quad f_{xx}=6x$
$f_{xy}(1;-3)=1 \quad f_{xy}=1$
$f_{yx}(1;-3)=1 \quad f_{yx}=1$
$f_{yy}(1;-3)=0 \quad f_{yy}=0$

$$\mathrm{d}^2f(1;-3)=6\cdot(x-1)\cdot(x-1)+1\cdot(x-1)\cdot(y+3)+1\cdot(y+3)\cdot(x-1)+ \\ +0\cdot(y+3)\cdot(y+3)=6(x-1)^2+2(y+3)$$

$f_{xxx}(1;-3)=6 \quad f_{xxx}=6$
$f_{xxy}(1;-3)=0 \quad f_{xxy}=0$
$f_{xyx}(1;-3)=0 \quad f_{xyx}=0$
$f_{xyy}(1;-3)=0 \quad f_{xyy}=0$
$f_{yxx}(1;-3)=0 \quad f_{yxx}=0$
$f_{yxy}(1;-3)=0 \quad f_{yxy}=0$
$f_{yyx}(1;-3)=0 \quad f_{yyx}=0$
$f_{yyy}(1;-3)=0 \quad f_{yyy}=0$

$\mathrm{d}^3f(1;-3)=6\cdot(x-1)\cdot(x-1)\cdot(x-1)+0+0+0+0+0+0+0=6(x-1)^3$

Všechny další parciální derivace jsou rovny nule,

$\mathrm{d}^4f(1;-3)=0;\ \mathrm{d}^5f(1;-3)=0;\ \mathrm{d}^6f(1;-3)=0;\ \mathrm{d}^7f(1;-3)=0;\ \ldots$ proto

$$\underline{\underline{T(E)=}}\quad f(1;-3)+\frac{\mathrm{d}f(1;-3)}{1!}+\frac{\mathrm{d}^2f(1;-3)}{2!}+\frac{\mathrm{d}^3f(1;-3)}{3!}=$$
$$\underline{\underline{3(x-1)^2+(x-1)(y+3)+(x-1)^3}}$$

13. Rozviňte do **Maclaurinovy** řady funkci

$$f(x,y) = \sin(x^2+y^2)\,. \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$f(0;0) = \sin(0^2+0^2) = \sin 0 = 0$

$f_x(0;0) = 0 \quad f_x = \cos(x^2+y^2)\cdot 2x = 2x\cdot\cos(x^2+y^2)$
$f_y(0;0) = 0 \quad f_y = \cos(x^2+y^2)\cdot 2y = 2y\cdot\cos(x^2+y^2)$

$\mathrm{d}f(0;0) = 0\cdot(x-0) + 0\cdot(y-0) = 0\cdot x + 0\cdot y = 0$

$f_{xx}(0;0) = 2 \quad f_{xx} = 2\cdot\cos(x^2+y^2) - 2x\cdot\sin(x^2+y^2)\cdot 2x =$
$= 2\cdot\cos(x^2+y^2) - 4x^2\cdot\sin(x^2+y^2)$
$f_{xy}(0;0) = 0 \quad f_{xy} = -2x\cdot\sin(x^2+y^2)\cdot 2y = -4xy\cdot\sin(x^2+y^2)$
$f_{yx}(0;0) = 0 \quad f_{yx} = -2y\cdot\sin(x^2+y^2)\cdot 2x = -4xy\cdot\sin(x^2+y^2)$
$f_{yy}(0;0) = 2 \quad f_{yy} = 2\cdot\cos(x^2+y^2) - 2y\cdot\sin(x^2+y^2)\cdot 2y =$
$= 2\cdot\cos(x^2+y^2) - 4y^2\cdot\sin(x^2+y^2)$

$$\mathrm{d}^2f(0;0) = 2\cdot x^2 + 0\cdot xy + 0\cdot yx + 2\cdot y^2 = 2\cdot(x^2+y^2) = \frac{2}{1!}\cdot(x^2+y^2)$$

$f_{xxx}(0;0) = 0 \quad f_{xxx} = -2\cdot\sin(x^2+y^2)\cdot 2x - 8x\cdot\sin(x^2+y^2) -$
$-4x^2\cdot\cos(x^2+y^2)\cdot 2x = -12x\cdot\sin(x^2+y^2) - 8x^3\cdot\cos(x^2+y^2)$
$f_{xxy}(0;0) = 0 \quad f_{xxy} = -2\cdot\sin(x^2+y^2)\cdot 2y - 4x^2\cdot\cos(x^2+y^2)\cdot 2y =$
$= -4y\cdot\sin(x^2+y^2) - 8x^2y\cdot\cos(x^2+y^2)$
$f_{xyx}(0;0) = 0 \quad f_{xyx} = -4y\cdot\sin(x^2+y^2) - 4xy\cdot\cos(x^2+y^2)\cdot 2x =$
$= -4y\cdot\sin(x^2+y^2) - 8x^2y\cdot\cos(x^2+y^2)$
$f_{xyy}(0;0) = 0 \quad f_{xyy} = -4x\cdot\sin(x^2+y^2) - 4xy\cdot\cos(x^2+y^2)\cdot 2y =$
$= -4x\cdot\sin(x^2+y^2) - 8xy^2\cdot\cos(x^2+y^2)$
$f_{yxx}(0;0) = 0 \quad f_{yxx} = -4y\cdot\sin(x^2+y^2) - 4xy\cdot\cos(x^2+y^2)\cdot 2x =$
$= -4y\cdot\sin(x^2+y^2) - 8x^2y\cdot\cos(x^2+y^2)$
$f_{yxy}(0;0) = 0 \quad f_{yxy} = -4x\cdot\sin(x^2+y^2) - 4xy\cdot\cos(x^2+y^2)\cdot 2y =$
$= -4x\cdot\sin(x^2+y^2) - 8xy^2\cdot\cos(x^2+y^2)$
$f_{yyx}(0;0) = 0 \quad f_{yyx} = -2\cdot\sin(x^2+y^2)\cdot 2x - 4y^2\cdot\cos(x^2+y^2)\cdot 2x =$
$= -4x\cdot\sin(x^2+y^2) - 8xy^2\cdot\cos(x^2+y^2)$
$f_{yyy}(0;0) = 0 \quad f_{yyy} = -2\cdot\sin(x^2+y^2)\cdot 2y - 8y\cdot\sin(x^2+y^2) -$
$-4y^2\cdot\cos(x^2+y^2)\cdot 2y = -12y\cdot\sin(x^2+y^2) - 8y^3\cdot\cos(x^2+y^2)$

$$\mathrm{d}^3f(0;0) = 0\cdot x^3 + 0\cdot x^2y + 0\cdot xyx + 0\cdot xy^2 + 0\cdot yx^2 + 0\cdot yxy + 0\cdot y^2x + 0\cdot y^3 = 0$$

$f_{xxxx}(0;0)=0 \quad f_{xxxx}=-12\cdot\sin(x^2+y^2)-12x\cdot\cos(x^2+y^2)\cdot 2x-$
$-24x^2\cdot\cos(x^2+y^2)+8x^3\cdot\sin(x^2+y^2)\cdot 2x=$
$=(16x^4-12)\cdot\sin(x^2+y^2)-48x^2\cdot\cos(x^2+y^2)$

$f_{xxxy}(0;0)=0 \quad f_{xxxy}=-12x\cdot\cos(x^2+y^2)\cdot 2y+8x^3\cdot\sin(x^2+y^2)\cdot 2y=$
$=16x^3y\cdot\sin(x^2+y^2)-24xy\cdot\cos(x^2+y^2)$

$f_{xxyx}(0;0)=0 \quad f_{xxyx}=-4y\cdot\cos(x^2+y^2)\cdot 2x-16xy\cdot\cos(x^2+y^2)+$
$+8x^2y\cdot\sin(x^2+y^2)\cdot 2x=16x^3y\cdot\sin(x^2+y^2)-24xy\cdot\cos(x^2+y^2)$

$f_{xxyy}(0;0)=0 \quad f_{xxyy}=-4\cdot\sin(x^2+y^2)-4y\cdot\cos(x^2+y^2)\cdot 2y-$
$-8x^2\cdot\cos(x^2+y^2)+8x^2y\sin(x^2+y^2)\cdot 2y=$
$=(16x^2y^2-4)\cdot\sin(x^2+y^2)+(-8x^2-8y^2)\cdot\cos(x^2+y^2)$

$f_{xyxx}(0;0)=0 \quad f_{xyxx}=-4y\cdot\cos(x^2+y^2)\cdot 2y-16xy\cdot\cos(x^2+y^2)+$
$+8x^2y\cdot\sin(x^2+y^2)\cdot 2x=16x^3y\cdot\sin(x^2+y^2)-24xy\cdot\cos(x^2+y^2)$

$f_{xyxy}(0;0)=0 \quad f_{xyxy}=-4\cdot\sin(x^2+y^2)-4y\cdot\cos(x^2+y^2)\cdot 2y-$
$-8x^2\cdot\cos(x^2+y^2)+8x^2y\cdot\sin(x^2+y^2)\cdot 2y=$
$=(16x^2y^2-4)\cdot\sin(x^2+y^2)+(-8x^2-8y^2)\cdot\cos(x^2+y^2)$

$f_{xyyx}(0;0)=0 \quad f_{xyyx}=-4\cdot\sin(x^2+y^2)-4x\cdot\cos(x^2+y^2)\cdot 2x-$
$-8y^2\cdot\cos(x^2+y^2)+8xy^2\cdot\sin(x^2+y^2)\cdot 2x=$
$=(16x^2y^2-4)\cdot\sin(x^2+y^2)+(-8x^2-8y^2)\cdot\cos(x^2+y^2)$

$f_{xyyy}(0;0)=0 \quad f_{xyyy}=-4x\cdot\cos(x^2+y^2)\cdot 2y-16xy\cdot\cos(x^2+y^2)+$
$+8xy^2\cdot\sin(x^2+y^2)\cdot 2y=16xy^3\cdot\sin(x^2+y^2)-24xy\cdot\cos(x^2+y^2)$

$f_{yxxx}(0;0)=0 \quad f_{yxxx}=-4y\cdot\cos(x^2+y^2)\cdot 2x-16xy\cdot\cos(x^2+y^2)+$
$+8x^2y\cdot\sin(x^2+y^2)\cdot 2x=16x^3y\cdot\sin(x^2+y^2)-24xy\cdot\cos(x^2+y^2)$

$f_{yxxy}(0;0)=0 \quad f_{yxxy}=-4\cdot\sin(x^2+y^2)-4y\cdot\cos(x^2+y^2)\cdot 2y-$
$-8x^2\cdot\cos(x^2+y^2)+8x^2y\cdot\sin(x^2+y^2)\cdot 2y=$
$=(16x^2y^2-4)\cdot\sin(x^2+y^2)+(-8x^2-8y^2)\cdot\cos(x^2+y^2)$

$f_{yxyx}(0;0)=0 \quad f_{yxyx}=-4\cdot\sin(x^2+y^2)-4x\cdot\cos(x^2+y^2)\cdot 2x-$
$-8y^2\cdot\cos(x^2+y^2)+8xy^2\cdot\sin(x^2+y^2)\cdot 2x=$
$=(16x^2y^2-4)\cdot\sin(x^2+y^2)+(-8x^2-8y^2)\cdot\cos(x^2+y^2)$

$f_{yxyy}(0;0)=0 \quad f_{yxyy}=-4x\cdot\cos(x^2+y^2)\cdot 2y-16xy\cos(x^2+y^2)+$
$+8xy^2\cdot\sin(x^2+y^2)\cdot 2y=16xy^3\cdot\sin(x^2+y^2)-24xy\cdot\cos(x^2+y^2)$

$f_{yyxx}(0;0)=0 \quad f_{yyxx}=-4\cdot\sin(x^2+y^2)-4x\cdot\cos(x^2+y^2)\cdot 2x-$
$-8y^2\cdot\cos(x^2+y^2)+8xy^2\cdot\sin(x^2+y^2)\cdot 2x=$
$=(16x^2y^2-4)\cdot\sin(x^2+y^2)+(-8x^2-8y^2)\cdot\cos(x^2+y^2)$

$f_{yyxy}(0;0)=0 \quad f_{yyxy}=-4x\cdot\cos(x^2+y^2)\cdot 2y-16xy\cdot\cos(x^2+y^2)+$
$+8xy^2\cdot\sin(x^2+y^2)\cdot 2y=16xy^3\cdot\sin(x^2+y^2)-24xy\cdot\cos(x^2+y^2)$

$f_{yyyx}(0;0)=0 \quad f_{yyyx}=-12y\cdot\cos(x^2+y^2)\cdot 2x+8y^3\cdot\sin(x^2+y^2)\cdot 2x=$
$=16xy^3\cdot\sin(x^2+y^2)-24xy\cdot\cos(x^2+y^2)$

$f_{yyyy}(0;0)=0 \quad f_{yyyy}=-12\cdot\sin(x^2+y^2)-12y\cdot\cos(x^2+y^2)\cdot 2y-$
$-24y^2\cdot\cos(x^2+y^2)+8y^3\cdot\sin(x^2+y^2)\cdot 2y=$
$=(16y^4-12)\cdot\sin(x^2+y^2)-48y^2\cos(x^2+y^2)$

$$d^4f(0;0)=0\cdot x^4+4\cdot 0\cdot x^3y+6\cdot 0\cdot x^2y^2+4\cdot 0\cdot xy^3+0\cdot y^4=0$$

$$f_{xxxxx}(0;0)=0 \quad f_{xxxxx} = 64x^3\cdot\sin(x^2+y^2)+(16x^4-12)\cdot\cos(x^2+y^2)\cdot 2x- -96x\cdot\cos(x^2+y^2)+48x^2\cdot\sin(x^2+y^2)\cdot 2x = = 160x^3\cdot\sin(x^2+y^2)+(32x^5-120x)\cdot\cos(x^2+y^2)$$

$$f_{xxxxy}(0;0)=0 \quad \ldots$$

$\vdots$

$$\mathrm{d}^5 f(0;0) = 0\cdot x^5 + 0\cdot x^4 y + \ldots = 0$$

$$f_{xxxxxx}(0;0)=-120 \quad f_{xxxxxx} = 480x^2\cdot\sin(x^2+y^2)+160x^3\cdot\cos(x^2+y^2)\cdot 2x+ +(160x^4-120)\cdot\cos(x^2+y^2)- -(32x^5-120x)\sin(x^2+y^2)\cdot 2x$$

$$f_{xxxxxy}(0;0)=0 \qquad \ldots$$

$\vdots$

$$\mathrm{d}^6 f(0;0) = -120\cdot x^6 + 0\cdot x^5\cdot y + \ldots - 40\cdot x^4\cdot y^2 + \ldots - 40\cdot x^3\cdot y\cdot x\cdot y + \ldots - 120\cdot y^6 =$$
$$= -120x^6 - 360x^4y^2 - 360x^2y^4 - 120y^6 = -120\cdot(x^6+3x^4y^2+3x^2y^4+y^6) =$$
$$= -\underbrace{6\cdot 5\cdot 4}_{120}\cdot\frac{3!}{3!}\cdot(x^2+y^2)^3$$

$\vdots$

$$\underline{\underline{M=}} \quad f(0;0)+\frac{\mathrm{d}f(0;0)}{1!}+\frac{\mathrm{d}^2f(0;0)}{2!}+\frac{\mathrm{d}^3f(0;0)}{3!}+\cdots=$$
$$=0+0+\frac{2!\cdot\frac{(x^2+y^2)}{1!}}{2!}+0+0+0+\frac{-6!\cdot\frac{(x^2+y^2)^3}{3!}}{6!}+0+\ldots=$$
$$\underline{\underline{=\frac{(x^2+y^2)}{1!}-\frac{(x^2+y^2)^3}{3!}+\frac{(x^2+y^2)^5}{5!}-\frac{(x^2+y^2)^7}{7!}+\frac{(x^2+y^2)^9}{9!}-\cdots}}$$

Poznámka

Rozvineme pro $t=0$ funkci (jedné proměnné) $\sin t$ a do výsledku dosadíme za $t = x^2+y^2$.

f	$=\sin t$	$f(0)$	$=$	0
f'	$=\cos t$	$f'(0)$	$=$	1
f''	$=-\sin t$	$f''(0)$	$=$	0
$f^{(3)}$	$=-\cos t$	$f^{(3)}(0)$	$=$	-1
$f^{(4)}$	$=\sin t$	$f^{(4)}(0)$	$=$	0
$f^{(5)}$	$=\cos t$	$f^{(5)}(0)$	$=$	1
$f^{(6)}$	$=-\sin t$	$f^{(6)}(0)$	$=$	0
$f^{(7)}$	$=-\cos t$	$t^{(7)}(0)$	$=$	-1
		$\vdots$		

$$\sin t = \frac{t}{1!}-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\frac{t^9}{9!}-\cdots$$