Funkce více proměnných

Parciální derivace funkce o více proměnných je její derivace vzhledem k jedné z těchto proměnných (derivujeme tuto funkci jako funkci jen ***jedné*** proměnné), kdy všechny ostatní proměnné pokládáme za konstanty.

Výsledkem parciální derivace je funkce. ***Tuto můžeme opět*** parciálně derivovat a dostáváme *parciální derivaci parciální derivace*, nebo-li druhou parciální derivaci (což je opět funkce). ***Tuto můžeme opět*** … třetí parciální derivaci …

1. Napište všechny třetí parciální derivace funkce

$$f(x,y) = x^3 + y^2 - 6xy - 39x + 18y + 4 \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$$f_x = \frac{\partial f(x,y)}{\partial x} = (x^3)' + 0 - 6y\,(x)' - 39\,(x)' + 0 + 0$$

$$f_y = \frac{\partial f(x,y)}{\partial y} = 0 + (y^2)' - 6x\,(y)' - 0 + 18\,(y)' + 0$$

$f_x = 3x^2 - 6y - 39$

- $f_{xx} = 6x$: $f_{xxx} = 6$, $f_{xxy} = 0$
- $f_{xy} = -6$: $f_{xyx} = 0$, $f_{xyy} = 0$

Jsou-li ***smíšené derivace*** f_{xy} a f_{yx} v okolí nějakého bodu $\mathcal{A}$ ***spojité***, pak jsou si v tomto bodě $\mathcal{A}$ rovny. Nebo-li: $f_{xy}(\mathcal{A}) = f_{yx}(\mathcal{A})$.

$f_y = 2y - 6x + 18$

- $f_{yx} = -6$: $f_{yxx} = 0$, $f_{yxy} = 0$
- $f_{yy} = 2$: $f_{yyx} = 0$, $f_{yyy} = 0$

Jsou-li ***smíšené derivace*** f_{xxy}, f_{xyx} a f_{yxx} v okolí bodu $\mathcal{A}$ ***spojité***, pak jsou si v tomto bodě $\mathcal{A}$ rovny.

$$f_{xxy}(\mathcal{A}) = f_{xyx}(\mathcal{A}) = f_{yxx}(\mathcal{A})$$

Podobně $f_{xyy}(\mathcal{A}), f_{yxy}(\mathcal{A}), f_{yyx}(\mathcal{A})$

2. Napište všechny třetí parciální derivace funkce

$$f(x,y) = 6xy^2 - 2x^3 - 3y^3 \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$f_x = 6y^2 - 6x^2$
$f_y = 12xy - 9y^2$

$f_{xx} = -12x$
$f_{xy} = 12y$
$f_{yx} = 12y$
$f_{yy} = 12x - 18y$

$f_{xxx} = -12$
$f_{xxy} = 0$
$f_{xyx} = 0$
$f_{xyy} = 12$
$f_{yxx} = 0$
$f_{yxy} = 12$
$f_{yyx} = 12$
$f_{yyy} = -18$

3. Napište všechny čtvrté parciální derivace funkce

$$f(x,y) = \mathrm{e}^{xy} \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$f_x = \mathrm{e}^{xy} \cdot y = y \cdot \mathrm{e}^{xy}$
$f_y = \mathrm{e}^{xy} \cdot x = x \cdot \mathrm{e}^{xy}$

$f_{xx} = y \cdot \mathrm{e}^{xy} \cdot y = y^2 \cdot \mathrm{e}^{xy}$
$f_{xy} = 1 \cdot \mathrm{e}^{xy} + y \cdot \mathrm{e}^{xy} \cdot x = (1 + xy) \cdot \mathrm{e}^{xy}$
$f_{yx} = 1 \cdot \mathrm{e}^{xy} + x \cdot \mathrm{e}^{xy} \cdot y = (1 + xy) \cdot \mathrm{e}^{xy}$
$f_{yy} = x \cdot \mathrm{e}^{xy} \cdot x = x^2 \cdot \mathrm{e}^{xy}$

Jsou-li ***smíšené derivace*** f_{xy} a f_{yx} v okolí bodu $\mathcal{A}$ ***spojité***, pak jsou si v tomto bodě $\mathcal{A}$ rovny.

Nebo-li: $f_{xy}(\mathcal{A}) = f_{yx}(\mathcal{A})$.

$f_{xxx} = y^2 \cdot \mathrm{e}^{xy} \cdot y = y^3 \cdot \mathrm{e}^{xy}$
$f_{xxy} = 2y \cdot \mathrm{e}^{xy} + y^2 \cdot \mathrm{e}^{xy} \cdot x = (2y + xy^2) \cdot \mathrm{e}^{xy}$
$f_{xyx} = y \cdot \mathrm{e}^{xy} + (1 + xy) \cdot \mathrm{e}^{xy} \cdot y = (2y + xy^2) \cdot \mathrm{e}^{xy}$
$f_{xyy} = x \cdot \mathrm{e}^{xy} + (1 + xy) \cdot \mathrm{e}^{xy} \cdot x = (2x + x^2y) \cdot \mathrm{e}^{xy}$
$f_{yxx} = y \cdot \mathrm{e}^{xy} + (1 + xy) \cdot \mathrm{e}^{xy} \cdot y = (2y + xy^2) \cdot \mathrm{e}^{xy}$
$f_{yxy} = x \cdot \mathrm{e}^{xy} + (1 + xy) \cdot \mathrm{e}^{xy} \cdot x = (2x + x^2y) \cdot \mathrm{e}^{xy}$
$f_{yyx} = 2x \cdot \mathrm{e}^{xy} + x^2 \cdot \mathrm{e}^{xy} \cdot y = (2x + x^2y) \cdot \mathrm{e}^{xy}$
$f_{yyy} = x^2 \cdot \mathrm{e}^{xy} \cdot x = x^3 \cdot \mathrm{e}^{xy}$

Jsou-li ***smíšené derivace*** f_{xxy}, f_{xyx} a f_{yxx} ... ***spojité***, pak jsou si v tomto bodě rovny.

Nebo-li: $f_{xxy}(\mathcal{A}) = f_{xyx}(\mathcal{A}) = f_{yxx}(\mathcal{A})$; $f_{xyy}(\mathcal{A}) = \dots$.

$f_{xxxx} = y^3 \cdot \mathrm{e}^{xy} \cdot y = y^4 \cdot \mathrm{e}^{xy}$
$f_{xxxy} = 3y^2 \cdot \mathrm{e}^{xy} + y^3 \cdot \mathrm{e}^{xy} \cdot x = (3y^2 + xy^3) \cdot \mathrm{e}^{xy}$
$f_{xxyx} = y^2 \cdot \mathrm{e}^{xy} + (2y + xy^2) \cdot \mathrm{e}^{xy} \cdot y = (3y^2 + xy^3) \cdot \mathrm{e}^{xy}$
$f_{xxyy} = (2 + 2xy) \cdot \mathrm{e}^{xy} + (2y + xy^2) \cdot \mathrm{e}^{xy} \cdot x = (2 + 4xy + x^2y^2) \cdot \mathrm{e}^{xy}$
$f_{xyxx} = y^2 \cdot \mathrm{e}^{xy} + (2y + xy^2) \cdot \mathrm{e}^{xy} \cdot y = (3y^2 + xy^3) \cdot \mathrm{e}^{xy}$
$f_{xyxy} = (2 + 2xy) \cdot \mathrm{e}^{xy} + (2y + xy^2) \cdot \mathrm{e}^{xy} \cdot x = (2 + 4xy + x^2y^2) \cdot \mathrm{e}^{xy}$
$f_{xyyx} = (2 + 2xy) \cdot \mathrm{e}^{xy} + (2x + x^2y) \cdot \mathrm{e}^{xy} \cdot y = (2 + 4xy + x^2y^2) \cdot \mathrm{e}^{xy}$
$f_{xyyy} = x^2 \cdot \mathrm{e}^{xy} + (2x + x^2y) \cdot \mathrm{e}^{xy} \cdot x = (3x^2 + x^3y) \cdot \mathrm{e}^{xy}$
$f_{yxxx} = y^2 \cdot \mathrm{e}^{xy} + (2y + xy^2) \cdot \mathrm{e}^{xy} \cdot y = (3y^2 + xy^3) \cdot \mathrm{e}^{xy}$
$f_{yxxy} = (2 + 2xy) \cdot \mathrm{e}^{xy} + (2y + xy^2) \cdot \mathrm{e}^{xy} \cdot x = (2 + 4xy + x^2y^2) \cdot \mathrm{e}^{xy}$
$f_{yxyx} = (2 + 2xy) \cdot \mathrm{e}^{xy} + (2x + x^2y) \cdot \mathrm{e}^{xy} \cdot y = (2 + 4xy + x^2y^2) \cdot \mathrm{e}^{xy}$
$f_{yxyy} = x^2 \cdot \mathrm{e}^{xy} + (2x + x^2y) \cdot \mathrm{e}^{xy} \cdot x = (3x^2 + x^3y) \cdot \mathrm{e}^{xy}$
$f_{yyxx} = (2 + 2xy) \cdot \mathrm{e}^{xy} + (2x + x^2y) \cdot \mathrm{e}^{xy} \cdot y = (2 + 4xy + x^2y^2) \cdot \mathrm{e}^{xy}$
$f_{yyxy} = x^2 \cdot \mathrm{e}^{xy} + (2x + x^2y) \cdot \mathrm{e}^{xy} \cdot x = (3x^2 + x^3y) \cdot \mathrm{e}^{xy}$
$f_{yyyx} = 3x^2 \cdot \mathrm{e}^{xy} + x^3 \cdot \mathrm{e}^{xy} \cdot y = (3x^2 + x^3y) \cdot \mathrm{e}^{xy}$
$f_{yyyy} = x^3 \cdot \mathrm{e}^{xy} \cdot x = x^4 \cdot \mathrm{e}^{xy}$

Jsou-li ***smíšené derivace*** $f_{xxxy}, f_{xxyx}, \dots$

4. Napište všechny čtvrté parciální derivace funkce

$$f(x,y) = \cos(xy) \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$$f_x = -\sin(xy)\cdot y = -y\cdot\sin(xy)$$
$$f_y = -\sin(xy)\cdot x = -x\cdot\sin(xy)$$

$$f_{xx} = -y\cdot\cos(xy)\cdot y = -y^2\cdot\cos(xy)$$
$$f_{xy} = -1\cdot\sin(xy) - y\cdot\cos(xy)\cdot x = -\sin(xy) - xy\cdot\cos(xy)$$
$$f_{yx} = -1\cdot\sin(xy) - x\cdot\cos(xy)\cdot y = -\sin(xy) - xy\cdot\cos(xy)$$
$$f_{yy} = -x\cdot\cos(xy)\cdot x = -x^2\cdot\cos(xy)$$

$$f_{xxx} = -y^2\cdot[-\sin(xy)\cdot y] = y^3\cdot\sin(xy)$$
$$f_{xxy} = -2y\cdot\cos(xy) - y^2\cdot[-\sin(xy)\cdot x] = xy^2\cdot\sin(xy) - 2y\cdot\cos(xy)$$
$$f_{xyx} = -\cos(xy).y - y.\cos(xy) + xy.\sin(xy).y = xy^2\cdot\sin(xy) - 2y\cdot\cos(xy)$$
$$f_{xyy} = -\cos(xy).x - x.\cos(xy) + xy.\sin(xy).x = x^2y\cdot\sin(xy) - 2x\cdot\cos(xy)$$
$$f_{yxx} = -\cos(xy).y - y.\cos(xy) + xy.\sin(xy).y = xy^2\cdot\sin(xy) - 2y\cdot\cos(xy)$$
$$f_{yxy} = -\cos(xy).x - x.\cos(xy) + xy.\sin(xy).x = x^2y\cdot\sin(xy) - 2x\cdot\cos(xy)$$
$$f_{yyx} = -2x\cdot\cos(xy) - x^2\cdot[-\sin(xy)\cdot y] = x^2y\cdot\sin(xy) - 2x\cdot\cos(xy)$$
$$f_{yyy} = -x^2\cdot[-\sin(xy)\cdot x] = x^3\cdot\sin(xy)$$

$$f_{xxxx} = y^3\cdot\cos(xy)\cdot y = y^4\cdot\cos(xy)$$
$$f_{xxxy} = 3y^2\cdot\sin(xy) + y^3\cdot\cos(xy)\cdot x = 3y^2\cdot\sin(xy) + xy^3\cdot\cos(xy)$$
$$f_{xxyx} = y^2.\sin(xy) + xy^2.\cos(xy).y + 2y.\sin(xy).y = 3y^2\cdot\sin(xy) + xy^3\cdot\cos(xy)$$
$$f_{xxyy} = 2xy\cdot\sin(xy) + xy^2\cdot\cos(xy)\cdot x - 2\cdot\cos(xy) - 2y\cdot[-\sin(xy)\cdot x] = 4xy\cdot\sin(xy) + (x^2y^2-2)\cdot\cos(xy)$$
$$f_{xyxx} = y^2.\sin(xy) + xy^2.\cos(xy).y + 2y.\sin(xy).y = 3y^2\cdot\sin(xy) + xy^3\cdot\cos(xy)$$
$$f_{xyxy} = 2xy\cdot\sin(xy) + xy^2\cdot\cos(xy)\cdot x - 2\cdot\cos(xy) - 2y\cdot[-\sin(xy)\cdot x] = 4xy\cdot\sin(xy) + (x^2y^2-2)\cdot\cos(xy)$$
$$f_{xyyx} = 2xy\cdot\sin(xy) + x^2y\cdot\cos(xy)\cdot y - 2\cdot\cos(xy) - 2x\cdot[-\sin(xy)\cdot y] = 4xy\cdot\sin(xy) + (x^2y^2-2)\cdot\cos(xy)$$
$$f_{xyyy} = x^2.\sin(xy) + x^2y.\cos(xy)\cdot x + 2x.\sin(xy).x = 3x^2\cdot\sin(xy) + x^3y\cdot\cos(xy)$$
$$f_{yxxx} = y^2.\sin(xy) + xy^2.\cos(xy).y + 2y.\sin(xy).y = 3y^2\cdot\sin(xy) + xy^3\cdot\cos(xy)$$
$$f_{yxxy} = 2xy\cdot\sin(xy) + xy^2\cdot\cos(xy)\cdot x - 2\cdot\cos(xy) - 2y\cdot[-\sin(xy)\cdot x] = 4xy\cdot\sin(xy) + (x^2y^2-2)\cdot\cos(xy)$$
$$f_{yxyx} = 2xy\cdot\sin(xy) + x^2y\cdot\cos(xy)\cdot y - 2\cdot\cos(xy) - 2x\cdot[-\sin(xy)\cdot y] = 4xy\cdot\sin(xy) + (x^2y^2-2)\cdot\cos(xy)$$
$$f_{yxyy} = x^2.\sin(xy) + x^2y.\cos(xy)\cdot x + 2x.\sin(xy).x = 3x^2\cdot\sin(xy) + x^3y\cdot\cos(xy)$$
$$f_{yyxx} = 2xy\cdot\sin(xy) + x^2y\cdot\cos(xy)\cdot y - 2\cdot\cos(xy) - 2x\cdot[-\sin(xy)\cdot y] = 4xy\cdot\sin(xy) + (x^2y^2-2)\cdot\cos(xy)$$
$$f_{yyxy} = x^2.\sin(xy) + x^2y.\cos(xy)\cdot x + 2x.\sin(xy).x = 3x^2\cdot\sin(xy) + x^3y\cdot\cos(xy)$$
$$f_{yyyx} = 3x^2\cdot\sin(xy) + x^3\cdot\cos(xy)\cdot y = 3x^2\cdot\sin(xy) + x^3y\cdot\cos(xy)$$
$$f_{yyyy} = x^3\cdot\cos(xy)\cdot x = x^4\cdot\cos(xy)$$

5. Napište všechny třetí parciální derivace funkce

$$f(x,y) = \sin(x^2 + y^2) \qquad [x \in \mathbb{R};\ y \in \mathbb{R}]$$

$f_x = \cos(x^2 + y^2) \cdot 2x = 2x \cdot \cos(x^2 + y^2)$

$f_y = \cos(x^2 + y^2) \cdot 2y = 2y \cdot \cos(x^2 + y^2)$

$f_{xx} = 2.\cos(x^2 + y^2) - 2x.\sin(x^2 + y^2).2x = 2 \cdot \cos(x^2 + y^2) - 4x^2 \cdot \sin(x^2 + y^2)$

$f_{xy} = -2x \cdot \sin(x^2 + y^2) \cdot 2y = -4xy \cdot \sin(x^2 + y^2)$

$f_{yx} = -2y \cdot \sin(x^2 + y^2) \cdot 2x = -4xy \cdot \sin(x^2 + y^2)$

$f_{yy} = 2.\cos(x^2 + y^2) - 2y.\sin(x^2 + y^2).2y = 2 \cdot \cos(x^2 + y^2) - 4y^2 \cdot \sin(x^2 + y^2)$

$f_{xxx} = -2 \cdot \sin(x^2 + y^2) \cdot 2x - 8x \cdot \sin(x^2 + y^2) - 4x^2 \cdot \cos(x^2 + y^2) \cdot 2x =$
$= -12x \cdot \sin(x^2 + y^2) - 8x^3 \cdot \cos(x^2 + y^2)$

$f_{xxy} = -2 \cdot \sin(x^2 + y^2) \cdot 2y - 4x^2 \cdot \cos(x^2 + y^2) \cdot 2y =$
$= -4y \cdot \sin(x^2 + y^2) - 8x^2y \cdot \cos(x^2 + y^2)$

$f_{xyx} = -4y \cdot \sin(x^2 + y^2) - 4xy \cdot \cos(x^2 + y^2) \cdot 2x =$
$= -4y \cdot \sin(x^2 + y^2) - 8x^2y \cdot \cos(x^2 + y^2)$

$f_{xyy} = -4x \cdot \sin(x^2 + y^2) - 4xy \cdot \cos(x^2 + y^2) \cdot 2y =$
$= -4x \cdot \sin(x^2 + y^2) - 8xy^2 \cdot \cos(x^2 + y^2)$

$f_{yxx} = -4y \cdot \sin(x^2 + y^2) - 4xy \cdot \cos(x^2 + y^2) \cdot 2x =$
$= -4y \cdot \sin(x^2 + y^2) - 8x^2y \cdot \cos(x^2 + y^2)$

$f_{yxy} = -4x \cdot \sin(x^2 + y^2) - 4xy \cdot \cos(x^2 + y^2) \cdot 2y =$
$= -4x \cdot \sin(x^2 + y^2) - 8xy^2 \cdot \cos(x^2 + y^2)$

$f_{yyx} = -2 \cdot \sin(x^2 + y^2) \cdot 2x - 4y^2 \cdot \cos(x^2 + y^2) \cdot 2x =$
$= -4x \cdot \sin(x^2 + y^2) - 8xy^2 \cdot \cos(x^2 + y^2)$

$f_{yyy} = -2 \cdot \sin(x^2 + y^2) \cdot 2y - 8y \cdot \sin(x^2 + y^2) - 4y^2 \cdot \cos(x^2 + y^2) \cdot 2y =$
$= -12y \cdot \sin(x^2 + y^2) - 8y^3 \cdot \cos(x^2 + y^2)$

6. Napište všechny druhé parciální derivace funkce

$$f(x,y,z) = (y^2 + 3z) \cdot \sin(4xz) \qquad [x \in \mathbb{R};\ y \in \mathbb{R};\ z \in \mathbb{R}]$$

$f_x = (y^2 + 3z) \cdot [\cos(4xz) \cdot 4z] = (4y^2 z + 12z^2) \cdot \cos(4xz)$

$f_y = 2y \cdot \sin(4xz)$

$f_z = 3.\sin(4xz) + (y^2 + 3z).[\cos(4xz).4x] = 3\sin(4xz) + (4xy^2 + 12xz) \cdot \cos(4xz)$

$f_{xx} = (4y^2 z + 12z^2) \cdot [-\sin(4xz) \cdot 4z]$
$f_{xy} = 8yz \cdot \cos(4xz)$
$f_{xz} = (4y^2 + 24z) \cdot \cos(4xz) + (4y^2 z + 12z^2) \cdot [-\sin(4xz) \cdot 4x]$

$f_{yx} = 2y \cdot [\cos(4xz) \cdot 4z]$
$f_{yy} = 2 \cdot \sin(4xz)$
$f_{yz} = 2y \cdot [\cos(4xz) \cdot 4x]$

$f_{zx} = 3 \cdot [\cos(4xz) \cdot 4z] + (4y^2 + 12z) \cdot \cos(4xz) + (4xy^2 + 12xz) \cdot [-\sin(4xz) \cdot 4z]$
$f_{zy} = 8xy \cdot \cos(4xz)$
$f_{zz} = 3 \cdot [\cos(4xz) \cdot 4x] + 12x \cdot \cos(4xz) + (4xy^2 + 12xz) \cdot [-\sin(4xz) \cdot 4x]$

7. Napište všechny druhé parciální derivace funkce

$$f(x,y,z) = (z^2 + 3y) \cdot \cos(4xy) \qquad [x \in \mathbb{R};\ y \in \mathbb{R};\ z \in \mathbb{R}]$$

$f_x = (z^2 + 3y) \cdot [-\sin(4xy) \cdot 4y] = (-4yz^2 - 12y^2) \cdot \sin(4xy)$

$f_y = 3\cos(4xy) + (z^2 + 3y).[-\sin(4xy).4x] = (-4xz^2 - 12xy).\sin(4xy) + 3.\cos(4xy)$

$f_z = 2z \cdot \cos(4xy)$

$f_{xx} = (-4yz^2 - 12y^2) \cdot [\cos(4xy) \cdot 4y]$
$f_{xy} = (-4z^2 - 24y) \cdot \sin(4xy) + (-4yz^2 - 12y^2) \cdot [\cos(4xy) \cdot 4x]$
$f_{xz} = -8yz \cdot \sin(4xy)$

$f_{yx} = (-4z^2 - 12y) \cdot \sin(4xy) + (-4xz^2 - 12xy) \cdot [\cos(4xy).4y] + 3 \cdot [-\sin(4xy).4y]$
$f_{yy} = -12x \cdot \sin(4xy) + (-4xz^2 - 12xy) \cdot [\cos(4xy) \cdot 4x] + 3 \cdot [-\sin(4xy) \cdot 4x]$
$f_{yz} = -8xz \cdot \sin(4xy)$

$f_{zx} = 2z \cdot [-\sin(4xy) \cdot 4y]$
$f_{zy} = 2z \cdot [-\sin(4xy) \cdot 4x]$
$f_{zz} = 2 \cdot \cos(4xy)$

8. Napište všechny druhé parciální derivace funkce

$$f(x,y,z) = (x^2 + 3y) \cdot \mathrm{e}^{4yz} \qquad [x \in \mathbb{R};\ y \in \mathbb{R};\ z \in \mathbb{R}]$$

$f_x = 2x \cdot \mathrm{e}^{4yz}$

$f_y = 3 \cdot \mathrm{e}^{4yz} + (x^2 + 3y) \cdot [\mathrm{e}^{4yz} \cdot 4z] = (3 + 4x^2z + 12yz) \cdot \mathrm{e}^{4yz}$

$f_z = (x^2 + 3y) \cdot [\mathrm{e}^{4yz} \cdot 4y] = (4x^2y + 12y^2) \cdot \mathrm{e}^{4yz}$

$f_{xx} = 2 \cdot \mathrm{e}^{4yz}$

$f_{xy} = 2x \cdot [\mathrm{e}^{4yz} \cdot 4z]$

$f_{xz} = 2x \cdot [\mathrm{e}^{4yz} \cdot 4y]$

$f_{yx} = 8xz \cdot \mathrm{e}^{4yz}$

$f_{yy} = 12z \cdot \mathrm{e}^{4yz} + (3 + 4x^2z + 12yz) \cdot [\mathrm{e}^{4yz} \cdot 4z]$

$f_{yz} = (4x^2 + 12y) \cdot \mathrm{e}^{4yz} + (3 + 4x^2z + 12yz) \cdot [\mathrm{e}^{4yz} \cdot 4y]$

$f_{zx} = 8xy \cdot \mathrm{e}^{4yz}$

$f_{zy} = (4x^2 + 24y) \cdot \mathrm{e}^{4yz} + (4x^2y + 12y^2) \cdot [\mathrm{e}^{4yz} \cdot 4z]$

$f_{zz} = (4x^2y + 12y^2) \cdot [\mathrm{e}^{4yz} \cdot 4y]$