(5) $-y'' + 3y = \sin x$, $y(0) = 1$, $y(1) = 2$, $h = \frac{1}{5}$.

Řešení!

$l = 1$, $h = \frac{1}{5} \Rightarrow n = \frac{l}{h} = \frac{1}{\frac{1}{5}} = 5 \Rightarrow x_i,\ i = 0, 1, \dots, 5$

$$x_i: \underset{x_0}{0}, \underset{x_1}{0.2}, \underset{x_2}{0.4}, \underset{x_3}{0.6}, \underset{x_4}{0.8}, \underset{x_5}{1}$$

- množina testovacích funkcí (prostor přípustných variací):

$$V = \{ v \in \mathcal{H}^1(0,1) : v(0) = v(1) = 0 \}$$

- množina přípustných řešení:

$$W = \{ w \in \mathcal{H}^1(0,1) : w(0) = 1,\ w(1) = 2 \}$$

$$-y'' + 3y = \sin x \qquad / \cdot v$$

$$-y''v + 3yv = \sin x \cdot v \qquad / \int_0^1 \dots dx$$

$$\int_0^1 (-y''v + 3yv)\,dx = \int_0^1 \sin x \cdot v\,dx$$

$$\int_0^1 (-y''v)\,dx + \int_0^1 3yv\,dx = \int_0^1 \sin x \cdot v\,dx \qquad (*)$$

per partes:

$$\int_0^1 (-y''v)\,dx = \begin{vmatrix} u = v & w' = -y'' \\ u' = v' & w = -y' \end{vmatrix} = [v \cdot (-y')]_0^1 + \int_0^1 v'y'\,dx = -\underbrace{v(1)}_{=0} y'(1) + \underbrace{v(0)}_{=0} y'(0) + \int_0^1 v'y'\,dx =$$

$= \int_0^1 v'y'\,dx$ ... dosadíme zpět do $(*)$ a dostáváme:

$$\underbrace{\int_0^1 v'y'\,dx + 3\int_0^1 yv\,dx}_{B(y,v)} = \underbrace{\int_0^1 \sin x \cdot v\,dx}_{L(v)}$$

- Galerkinova (slabá) formulace úlohy:

Najděte funkci $y \in W$ tak, aby bylo splněno $B(y,v) = L(v)\ \ \forall v \in V$

- tvar slabého řešení:

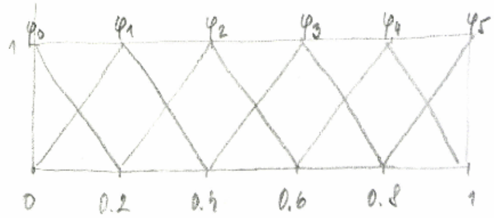
$$y_h(x) = \underbrace{y_0}_{1} \varphi_0(x) + y_1 \varphi_1(x) + y_2 \varphi_2(x) + y_3 \varphi_3(x) + y_4 \varphi_4(x) + \underbrace{y_5}_{2} \varphi_5(x) \quad \in W$$

$$y_h(x) = \varphi_0(x) + y_1 \varphi_1(x) + y_2 \varphi_2(x) + y_3 \varphi_3(x) + y_4 \varphi_4(x) + 2\varphi_5(x)$$

... 4 neznámé $y_1, y_2, y_3, y_4$

• maticový zápis soustavy lin. rovnic:

$$
\begin{pmatrix}
B(\varphi_1,\varphi_1) & B(\varphi_2,\varphi_1) & \overset{=0}{\underline{B(\varphi_3,\varphi_1)}} & \overset{=0}{\underline{B(\varphi_4,\varphi_1)}} \\
B(\varphi_1,\varphi_2) & B(\varphi_2,\varphi_2) & B(\varphi_3,\varphi_2) & \overset{=0}{\underline{B(\varphi_4,\varphi_2)}} \\
\overset{=0}{\underline{B(\varphi_1,\varphi_3)}} & B(\varphi_2,\varphi_3) & B(\varphi_3,\varphi_3) & B(\varphi_4,\varphi_3) \\
\underline{B(\varphi_1,\varphi_4)}=0 & \underline{B(\varphi_2,\varphi_4)}=0 & B(\varphi_3,\varphi_4) & B(\varphi_4,\varphi_4)
\end{pmatrix}
\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}
=
\begin{pmatrix} L(\varphi_1) - y_0 B(\varphi_0,\varphi_1) \\ L(\varphi_2) \\ L(\varphi_3) \\ L(\varphi_4) - y_5 B(\varphi_5,\varphi_4) \end{pmatrix}
$$




$$\varphi_0(x) = \begin{cases} -5\left(x-\frac{1}{5}\right) = 1-5x & 0 \le x \le \frac{1}{5} \\ 0 & \text{jinak} \end{cases} \qquad \varphi_0'(x) = \begin{cases} -5 & 0 \le x \le 0.2 \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_1(x) = \begin{cases} 5x & 0 \le x \le 0.2 \\ 2-5x & 0.2 \le x \le 0.4 \\ 0 & \text{jinak} \end{cases} \qquad \varphi_1'(x) = \begin{cases} 5 & 0 \le x \le 0.2 \\ -5 & 0.2 \le x \le 0.4 \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_2(x) = \begin{cases} 5x-1 & 0.2 \le x \le 0.4 \\ 3-5x & 0.4 \le x \le 0.6 \\ 0 & \text{jinak} \end{cases} \qquad \varphi_2'(x) = \begin{cases} 5 & 0.2 \le x \le 0.4 \\ -5 & 0.4 \le x \le 0.6 \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_3(x) = \begin{cases} 5x-2 & 0.4 \le x \le 0.6 \\ 4-5x & 0.6 \le x \le 0.8 \\ 0 & \text{jinak} \end{cases} \qquad \varphi_3'(x) = \begin{cases} 5 & 0.4 \le x \le 0.6 \\ -5 & 0.6 \le x \le 0.8 \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_4(x) = \begin{cases} 5x-3 & 0.6 \le x \le 0.8 \\ 5-5x & 0.8 \le x \le 1 \\ 0 & \text{jinak} \end{cases} \qquad \varphi_4'(x) = \begin{cases} 5 & 0.6 \le x \le 0.8 \\ -5 & 0.8 \le x \le 1 \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_5(x) = \begin{cases} 5x-4 & 0.8 \le x \le 1 \\ 0 & \text{jinak} \end{cases} \qquad \varphi_5'(x) = \begin{cases} 5 & 0.8 \le x \le 1 \\ 0 & \text{jinak} \end{cases}$$

$$\underbrace{\int_0^1 \varphi' v' dx + 3\int_0^1 \varphi v\, dx}_{B(\varphi, v)} = \underbrace{\int_0^1 \sin x \cdot v\, dx}_{L(v)}$$

$$B(\varphi_1, \varphi_1) = \int_0^{0.2} \left(5 \cdot 5 + 3 \cdot 5x \cdot 5x\right) dx + \int_{0.2}^{0.4} \left(-5 \cdot (-5) + 3 \cdot (2-5x)^2\right) dx =$$

$$= \int_0^{0.2} (25 + 75x^2)\, dx + \int_{0.2}^{0.4} \left(25 + 3 \cdot (4 - 20x + 25x^2)\right) dx = 10.4$$

$$B(\varphi_2, \varphi_2) = \int_{0.2}^{0.4} \left(5 \cdot 5 + 3 \cdot (5x-1)^2\right) dx + \int_{0.4}^{0.6} \left(-5 \cdot (-5) + 3 \cdot (3-5x)^2\right) dx = 10.4$$

$$B(\varphi_3, \varphi_3) = \int_{0.4}^{0.6} \left(5 \cdot 5 + 3(5x-2)^2\right) dx + \int_{0.6}^{0.8} \left(-5 \cdot (-5) + 3(4-5x)^2\right) dx = 10.4$$

$$B(\varphi_4, \varphi_4) = \int_{0.6}^{0.8} \left(5 \cdot 5 + 3(5x-3)^2\right) dx + \int_{0.8}^{1} \left(-5 \cdot (-5) + 3(5-5x)^2\right) dx = 10.4$$

$$B(\varphi_2, \varphi_1) = \int_{0.2}^{0.4} \left(5 \cdot (-5) + 3(5x-1)(2-5x)\right) dx = -4.9$$

$$B(\varphi_3, \varphi_2) = \int_{0.4}^{0.6} \left(5 \cdot (-5) + 3 \cdot (5x-2)(3-5x)\right) dx = -4.9$$

$$B(\varphi_4, \varphi_3) = \int_{0.6}^{0.8} \left(5 \cdot (-5) + 3 \cdot (5x-3)(4-5x)\right) dx = -4.9$$

$$B(\varphi_1, \varphi_2) = \int_{0.2}^{0.4} \left(-5 \cdot 5 + 3 \cdot (2-5x) \cdot (5x-1)\right) dx = -4.9$$

$$B(\varphi_2, \varphi_3) = \int_{0.4}^{0.6} \left(-5 \cdot 5 + 3(3-5x) \cdot (5x-2)\right) dx = -4.9$$

$$B(\varphi_3, \varphi_4) = \int_{0.6}^{0.8} \left(-5 \cdot 5 + 3 \cdot (4-5x) \cdot (5x-3)\right) dx = -4.9$$

$$L(\varphi_1): \int_0^{0.2} \sin x \cdot 5x\, dx = \left| \begin{array}{ll} u = 5x & v' = \sin x \\ u' = 5 & v = -\cos x \end{array} \right| = -5\left[x \cos x\right]_0^{0.2} + 5\int_0^{0.2} \cos x\, dx =$$

$$= -5(0.2 \cos 0.2 - 0) + 5\left[\sin x\right]_0^{0.2} =$$

$$= -\cos 0.2 + 5 \sin 0.2 = 0.0133$$

$$\int_{0.2}^{0.4} \sin x \cdot (2-5x)\, dx = 2\left[-\cos x\right]_{0.2}^{0.4} - 5\int_{0.2}^{0.4} x \sin x\, dx = -2(\cos 0.4 - \cos 0.2) -$$

$$-5\left(\left[-x \cos x\right]_{0.2}^{0.4} + \int_{0.2}^{0.4} \cos x\, dx\right) = -2\cos 0.4 + 2\cos 0.2 - 5(-0.4\cos 0.4 + 0.2 \cos 0.2) -$$

$$-5\left[\sin x\right]_{0.2}^{0.4} = \cancel{-4\cos 0.4} + \cos 0.2 - 5 \sin 0.4 + 5 \sin 0.2 = 0.0263$$

$$L(\varphi_1) = \int_0^{0.2} \sin x \cdot 5x + \int_{0.2}^{0.4} \sin x \cdot (2-5x)\, dx = 0.0133 + 0.0263 = 0.0396$$

$$B(\varphi_0, \varphi_1) = \int_0^{0.2} \left(-5 \cdot 5 + 3 \cdot (1-5x) \cdot (5x)\right) dx = -4.9$$

$$\tilde{L}(\varphi_1) = L(\varphi_1) - y_0 B(\varphi_0, \varphi_1) = 0.0396 - 1 \cdot (-4.9) = 4.9396$$

$$L(\varphi_2) = \int_{0.2}^{0.4} \sin x \cdot (5x-1)\, dx + \int_{0.4}^{0.6} \sin x \cdot (3-5x)\, dx = 10 \sin(0.4) - 5 \sin(0.2) + 5 \sin(0.6) = 0.0776$$

$$L(\varphi_3) = \int_{0.4}^{0.6} \sin x \cdot (5x-2)\, dx + \int_{0.6}^{0.8} \sin x \cdot (4-5x)\, dx = 10 \sin(0.6) - 5 \sin(0.4) + 5 \sin(0.8) = 0.1126$$

$$L(\varphi_4) = \int_{0.6}^{0.8} \sin x \cdot (5x-3)\, dx + \int_{0.8}^{1} \sin x \cdot (5-5x)\, dx = 10 \sin(0.8) - 5 \sin(0.6) + 5 \sin(1) = 0.1430$$

$$B(\varphi_5, \varphi_4) = \int_{0.8}^{1} \left(5 \cdot (-5) + 3 \cdot (5x-4) \cdot (5-5x)\right) dx = -4.9$$

$$\tilde{L}(\varphi_4) = L(\varphi_4) - y_5 B(\varphi_5, \varphi_4) = 0.1430 - 2 \cdot (-4.9) = 9.943$$

$$\Downarrow$$

- soustava lin. rovnic:

$$\begin{pmatrix} 10.4 & -4.9 & 0 & 0 \\ -4.9 & 10.4 & -4.9 & 0 \\ 0 & -4.9 & 10.4 & -4.9 \\ 0 & 0 & -4.9 & 10.4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 4.9396 \\ 0.0776 \\ 0.1126 \\ 9.943 \end{pmatrix}$$

- řešení soustavy:

$y_0 = 1$

$y_1 = 0.9632$

$y_2 = 1.0363$

$y_3 = 1.2205$

$y_4 = 1.5311$

$y_5 = 2$

Přibližné řešení diferenciální rovnice:
$y_h = \varphi_0 + 0.9632\, \varphi_1 + 1.0363\, \varphi_2 + 1.2205\, \varphi_3 + 1.5311\, \varphi_4 + 2\, \varphi_5$