

najděte aproximaci řešení úlohy $-0.4y'' + y' = 2$; $y(0) = -1$
 $-0.4y'(1) = 2y(1)$

s krokem $\frac{1}{4}$.

Řešení!

$$h=1, \quad h = \frac{1}{4} \Rightarrow n = \frac{b}{h} = \frac{1}{\frac{1}{4}} = 4 \Rightarrow i = 0, 1, 2, 3, 4$$

$$x_i = i \cdot h, \quad i = 0, \dots, 4$$

$$x_i: 0; \frac{1}{4}; \frac{1}{2}; \frac{3}{4}; 1$$

• prostor přípustných variací: $V = \{v \in \mathcal{C}^1(0,1) : v(0) = 0\}$

(ma' hodnotu 0 v těch hraničních bodech, v nichž je hodnota hledané funkce dána)

• množina přípustných řešení: $W = \{w \in \mathcal{C}^1(0,1) : w(0) = -1\}$

$$-0.4 y'' + y' = 2 \quad | \cdot v$$

$$-0.4 y'' v + y' v = 2v \quad | \int_0^1 \dots dx$$

$$\int_0^1 (-0.4 y'' v + y' v) dx = \int_0^1 2v dx$$

$$\underbrace{\int_0^1 (-0.4 y'' v) dx}_{\text{použijeme metodu per partes}} + \int_0^1 y' v dx = \int_0^1 2v dx \quad (*)$$

$$\int_0^1 (-0.4 y'' v) dx = \left| \begin{array}{ll} u = v & u' = -0.4 y'' \\ u' = v' & u = -0.4 y' \end{array} \right| = -0.4 [v \cdot y']_0^1 + 0.4 \int_0^1 v' y' dx =$$

$$= -0.4 \left[\underbrace{v(1) y'(1)}_{=0} - \underbrace{v(0) y'(0)}_{=0} \right] + 0.4 \int_0^1 v' y' dx = 0.4 \int_0^1 v' y' dx - 0.4 \cdot \frac{2}{-0.4} y(1) v(1) =$$

x podmínky $-0.4 y'(1) = 2 y(1)$

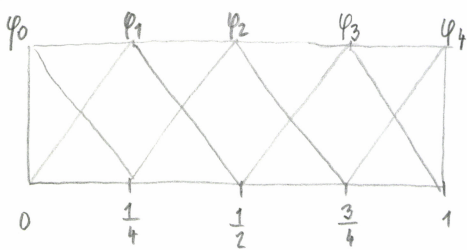
dostáváme: $y'(1) = \frac{2}{-0.4} y(1)$

$$= 0.4 \int_0^1 v' y' dx + 2 y(1) v(1) \quad \dots \text{dostáváme do } (*)$$

$$0.4 \int_0^1 y' v' dx + \int_0^1 y' v dx + 2 y(1) v(1) = \int_0^1 2v dx$$

$$\underbrace{\int_0^1 (0.4 y' v' + y' v) dx}_{B(y, v)} + \underbrace{2 y(1) v(1)}_{L(v)} = \int_0^1 2v dx$$

Galerkinova formulace: najděte $y \in W$ tak, aby bylo splněno $B(y, v) = L(v) \quad \forall v \in V$.



$$\varphi_0(x) = \begin{cases} -4x+1 & 0 \leq x \leq \frac{1}{4} \\ 0 & \text{final} \end{cases}$$

$$\varphi_1(x) = \begin{cases} 4x & 0 \leq x \leq \frac{1}{4} \\ -4x+2 & \frac{1}{4} \leq x \leq \frac{1}{2} \\ 0 & \text{final} \end{cases}$$

$$\varphi_2(x) = \begin{cases} 4x-1 & \frac{1}{4} \leq x \leq \frac{1}{2} \\ -4x+3 & \frac{1}{2} \leq x \leq \frac{3}{4} \\ 0 & \text{final} \end{cases}$$

$$\varphi_3(x) = \begin{cases} 4x-2 & \frac{1}{2} \leq x \leq \frac{3}{4} \\ -4x+4 & \frac{3}{4} \leq x \leq 1 \\ 0 & \text{final} \end{cases}$$

$$\varphi_4(x) = \begin{cases} 4x-3 & \frac{3}{4} \leq x \leq 1 \\ 0 & \text{final} \end{cases}$$

$$\varphi_0'(x) = \begin{cases} -4 & 0 \leq x \leq \frac{1}{4} \\ 0 & \text{final} \end{cases}$$

$$\varphi_1'(x) = \begin{cases} 4 & 0 \leq x \leq \frac{1}{4} \\ -4 & \frac{1}{4} \leq x \leq \frac{1}{2} \\ 0 & \text{final} \end{cases}$$

$$\varphi_2'(x) = \begin{cases} 4 & \frac{1}{4} \leq x \leq \frac{1}{2} \\ -4 & \frac{1}{2} \leq x \leq \frac{3}{4} \\ 0 & \text{final} \end{cases}$$

$$\varphi_3'(x) = \begin{cases} 4 & \frac{1}{2} \leq x \leq \frac{3}{4} \\ -4 & \frac{3}{4} \leq x \leq 1 \\ 0 & \text{final} \end{cases}$$

$$\varphi_4'(x) = \begin{cases} 4 & \frac{3}{4} \leq x \leq 1 \\ 0 & \text{final} \end{cases}$$

$$B(\varphi_0, \varphi_0) = \int_0^{0.25} (0.4 \cdot (-4) \cdot (-4) + (-4) \cdot (-4x+1)) dx = \int_0^{0.25} (6.4 + 16x - 4) dx = \int_0^{0.25} (2.4 + 16x) dx =$$

$$= 2.4 [x]_0^{0.25} - \frac{16}{2} [x^2]_0^{0.25} = 2.4 \cdot 0.25 - 8 \cdot 0.25^2 = \underline{0.1}$$

$$B(\varphi_1, \varphi_1) = \int_0^{0.25} (0.4 \cdot 4 \cdot 4 + 4 \cdot 4x) dx + \int_{0.25}^{0.5} (0.4 \cdot (-4) \cdot (-4) + (-4) \cdot (-4x+2)) dx =$$

$$= \int_0^{0.25} (6.4 + 16x) dx + \int_{0.25}^{0.5} (6.4 + 16x - 8) dx = 6.4 [x]_0^{0.25} + \frac{16}{2} [x^2]_0^{0.25} - 1.6 [x]_{0.25}^{0.5} + \frac{16}{2} [x^2]_{0.25}^{0.5} =$$

$$= 6.4 \cdot 0.25 + 8 \cdot 0.25^2 - 1.6 \cdot (0.5 - 0.25) + 8(0.5^2 - 0.25^2) = \underline{3.2}$$

$$B(\varphi_2, \varphi_2) = \int_0^{0.25} (0.4 \cdot 4 \cdot 4 + 4 \cdot (4x-1)) dx + \int_{0.25}^{0.45} (0.4 \cdot (-4) \cdot (-4) + (-4) \cdot (-4x+3)) dx =$$

$$= \int_0^{0.25} (6.4 + 16x - 4) dx + \int_{0.25}^{0.45} (6.4 + 16x - 12) dx = 2.4 [x]_{0.25}^{0.5} + \frac{16}{2} [x^2]_{0.25}^{0.5} - 5.6 [x]_{0.5}^{0.45} + \frac{16}{2} [x^2]_{0.5}^{0.45} = \underline{3.2}$$

$$B(\varphi_3, \varphi_3) = \int_0^{0.25} (0.4 \cdot 4 \cdot 4 + 4 \cdot (4x-2)) dx + \int_{0.45}^1 (0.4 \cdot (-4) \cdot (-4) + (-4) \cdot (-4x+4)) dx =$$

$$= \int_0^{0.25} (6.4 + 16x - 8) dx + \int_{0.45}^1 (6.4 + 16x - 16) dx = -1.6 [x]_{0.5}^{0.45} + \frac{16}{2} [x^2]_{0.5}^{0.45} - 9.6 [x]_{0.45}^1 + \frac{16}{2} [x^2]_{0.45}^1 = \underline{3.2}$$

$$B(\varphi_4, \varphi_4) = \int_{0.45}^1 (0.4 \cdot 4 \cdot 4 + 4 \cdot (4x-3)) dx + \underbrace{2}_{\varphi_4(1)} \underbrace{\varphi_4(1)}_{=1} = \int_{0.45}^1 (6.4 + 16x - 12) dx + 2 \cdot 1 \cdot 1 =$$

$$= -5.6 [x]_{0.45}^1 + \frac{16}{2} [x^2]_{0.45}^1 + 2 = \underline{4.1}$$

$$B(\varphi_0, \varphi_1) = \int_0^{0.25} (0.4 \cdot (-4) \cdot 4 + (-4) \cdot 4x) dx = \int_0^{0.25} (-6.4 - 16x) dx = -6.4 [x]_0^{0.25} - \frac{16}{2} [x^2]_0^{0.25} = \underline{-2.1}$$

$$B(\varphi_1, \varphi_2) = \int_{0.25}^{0.5} (0.4 \cdot (-4) \cdot 4 + (-4) \cdot (4x-1)) dx = \int_{0.25}^{0.5} (-6.4 - 16x + 4) dx = -2.4 [x]_{0.25}^{0.5} - \frac{16}{2} [x^2]_{0.25}^{0.5} = \underline{-2.1}$$

$$B(\varphi_2, \varphi_3) = \int_{0.5}^{0.45} (0.4 \cdot (-4) \cdot 4 + (-4) \cdot (4x-2)) dx = \int_{0.5}^{0.45} (-6.4 - 16x + 8) dx = 1.6 [x]_{0.5}^{0.45} - \frac{16}{2} [x^2]_{0.5}^{0.45} = \underline{-2.1}$$

$$B(\varphi_3, \varphi_4) = \int_{0.45}^1 (0.4 \cdot (-4) \cdot 4 + (-4) \cdot (4x-3)) dx = \int_{0.45}^1 (-6.4 - 16x + 12) dx = 5.6 [x]_{0.45}^1 - \frac{16}{2} [x^2]_{0.45}^1 = \underline{-2.1}$$

$$B(\varphi_1, \varphi_0) = \int_0^{0.25} (0.4 \cdot 4 \cdot (-4) + 4 \cdot (-4x+1)) dx = \int_0^{0.25} (-6.4 - 16x + 4) dx = -2.4 [x]_0^{0.25} - \frac{16}{2} [x^2]_0^{0.25} = \underline{-1.1}$$

$$B(\varphi_2, \varphi_1) = \int_{0.25}^{0.45} (0.4 \cdot 4 \cdot (-4) + 4 \cdot (-4x+2)) dx = \int_{0.25}^{0.45} (-6.4 - 16x + 8) dx = 1.6 [x]_{0.25}^{0.5} - \frac{16}{2} [x^2]_{0.25}^{0.5} = \underline{-1.1}$$

$$B(\varphi_3, \varphi_2) = \int_{0.5}^{0.45} (0.4 \cdot 4 \cdot (-4) + 4 \cdot (-4x+3)) dx = \int_{0.5}^{0.45} (-6.4 - 16x + 12) dx = 5.6 [x]_{0.5}^{0.45} - \frac{16}{2} [x^2]_{0.5}^{0.45} = \underline{-1.1}$$

$$B(\varphi_4, \varphi_3) = \int_{0.45}^1 (0.4 \cdot 4 \cdot (-4) + 4 \cdot (-4x+4)) dx = \int_{0.45}^1 (-6.4 - 16x + 16) dx = 9.6 [x]_{0.45}^1 - \frac{16}{2} [x^2]_{0.45}^1 = \underline{-1.1}$$

$$\text{Řešme } L(\varphi) = \int_0^1 2\varphi \, dx$$

$$L(\varphi_0) = \int_0^{0.25} 2 \cdot (-4x+1) \, dx = -\frac{2}{2} [x^2]_0^{0.25} + 2[x]_0^{0.25} = \underline{0.25}$$

$$L(\varphi_1) = \int_0^{0.25} 2 \cdot (4x) \, dx + \int_{0.25}^{0.5} 2 \cdot (-4x+2) \, dx = \frac{2}{2} [x^2]_0^{0.25} - \frac{2}{2} [x^2]_{0.25}^{0.5} + 4[x]_{0.25}^{0.5} = \underline{0.5}$$

$$L(\varphi_2) = \int_{0.25}^{0.5} 2 \cdot (4x-1) \, dx + \int_{0.5}^{0.75} 2 \cdot (-4x+3) \, dx = \frac{2}{2} [x^2]_{0.25}^{0.5} - 2[x]_{0.25}^{0.5} - \frac{2}{2} [x^2]_{0.5}^{0.75} + 6[x]_{0.5}^{0.75} = \underline{0.5}$$

$$L(\varphi_3) = \int_{0.5}^{0.75} 2 \cdot (4x-2) \, dx + \int_{0.75}^1 2 \cdot (-4x+4) \, dx = \frac{2}{2} [x^2]_{0.5}^{0.75} - 4[x]_{0.5}^{0.75} - \frac{2}{2} [x^2]_{0.75}^1 + 8[x]_{0.75}^1 = \underline{0.5}$$

$$L(\varphi_4) = \int_{0.75}^1 2 \cdot (4x-3) \, dx = \frac{2}{2} [x^2]_{0.75}^1 - 6[x]_{0.75}^1 = \underline{0.25}$$

Matricový zápis soustavy lineárních rovnic:

$$\begin{pmatrix} B(\varphi_0, \varphi_0) & B(\varphi_1, \varphi_0) & 0 & 0 & 0 \\ B(\varphi_0, \varphi_1) & B(\varphi_1, \varphi_1) & B(\varphi_2, \varphi_1) & 0 & 0 \\ 0 & B(\varphi_1, \varphi_2) & B(\varphi_2, \varphi_2) & B(\varphi_3, \varphi_2) & 0 \\ 0 & 0 & B(\varphi_2, \varphi_3) & B(\varphi_3, \varphi_3) & B(\varphi_4, \varphi_3) \\ 0 & 0 & 0 & B(\varphi_3, \varphi_4) & B(\varphi_4, \varphi_4) \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} L(\varphi_0) \\ L(\varphi_1) \\ L(\varphi_2) \\ L(\varphi_3) \\ L(\varphi_4) \end{pmatrix}$$

Každé uvážíme okrajovou podmínku $y(0) = -1$ (ke každé), tj. $y(0) = y_0 = -1 \Rightarrow$ první složku vektoru řešení $y = (y_0, y_1, \dots, y_4)^T$ zvolíme $\Rightarrow$ 1. rovnici můžeme vyzkoušet. Zbylé upravit 2. rovnici systému (rozeptat):

$$B(\varphi_0, \varphi_1) \cdot y_0 + B(\varphi_1, \varphi_1) \cdot y_1 + B(\varphi_2, \varphi_1) \cdot y_2 = L(\varphi_1)$$

$-1 \Rightarrow$ převeďme na pravou stranu a dostáváme:

$$B(\varphi_1, \varphi_1) \cdot y_1 + B(\varphi_2, \varphi_1) \cdot y_2 = L(\varphi_1) + B(\varphi_0, \varphi_1) = 0.5 - 2 \cdot 1 = -1.5$$

Celkem řešíme systém:

$$\begin{pmatrix} B(\varphi_1, \varphi_1) & B(\varphi_2, \varphi_1) & 0 & 0 \\ B(\varphi_1, \varphi_2) & B(\varphi_2, \varphi_2) & B(\varphi_3, \varphi_2) & 0 \\ 0 & B(\varphi_2, \varphi_3) & B(\varphi_3, \varphi_3) & B(\varphi_4, \varphi_3) \\ 0 & 0 & B(\varphi_3, \varphi_4) & B(\varphi_4, \varphi_4) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} L(\varphi_1) + B(\varphi_0, \varphi_1) \\ L(\varphi_2) \\ L(\varphi_3) \\ L(\varphi_4) \end{pmatrix}$$

díky okrajové podmínce

Do dosazení řešení systému lineárních rovnic:

$$\begin{pmatrix} 3.2 & -1.1 & 0 & 0 \\ -2.1 & 3.2 & -1.1 & 0 \\ 0 & -2.1 & 3.2 & -1.1 \\ 0 & 0 & -2.1 & 4.1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -1.6 \\ 0.5 \\ 0.5 \\ 0.25 \end{pmatrix}$$

$$\text{řešení } y = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -0.5643 \\ -0.1956 \\ 0.0593 \\ 0.0913 \end{pmatrix} \left. \begin{array}{l} \rightarrow \text{okrajové podmínky} \\ \text{řešení systému} \end{array} \right\}$$