Uvažujme úlohu

$$-a^2y'' + py' + qy = f \text{ pro } x \in (0, l) \tag{1}$$

a okrajové podmínky

- Dirichletovy: $y(0) = \alpha_0$, $y(l) = \alpha_l$,
- Neumannovy: $a^2y'(0) = \beta_0$, $-a^2y'(l) = \beta_l$,
- Newtonovy: $a^2y'(0) = \gamma_0 y(0) + \beta_0$, $-a^2y'(l) = \gamma_l y(l) + \beta_l$,

METODA KONEČNÝCH PRVKŮ

- prostor testovacích funkcí V
- množina přípustných řešení W
- Galerkinova (slabá) formulace úlohy: Najděte funkci $y \in W$ tak, aby

$$\begin{aligned} B(y,v) &= L(v) \quad \forall v \in V, \text{ kde} \\ B(y,v) &= \dots \\ L(v) &= \dots \\ V &= \dots \\ W &= \dots \end{aligned}$$

- diskretizece úlohy: $n = \frac{l}{h}$, l - délka intervalu, h - diskretizační krok
- přibližné řešení: $y_h = \sum\limits_{j=0}^{n} c_j \varphi_j$, neznámé koeficienty $c_0, c_1, \dots, c_n$ dostáváme jako řešení soustavy lineárních rovnic

$$K\boldsymbol{c} = \boldsymbol{b}, \text{ kde}$$

K se nazývá matice tuhosti a $\boldsymbol{b}$ vektor zatížení.

$$\begin{pmatrix} B(\varphi_0,\varphi_0) & B(\varphi_1,\varphi_0) & \cdots & B(\varphi_n,\varphi_0) \\ B(\varphi_0,\varphi_1) & B(\varphi_1,\varphi_1) & \cdots & B(\varphi_n,\varphi_1) \\ \vdots & & \ddots & \vdots \\ B(\varphi_0,\varphi_n) & B(\varphi_1,\varphi_n) & \cdots & B(\varphi_n,\varphi_n) \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} L(\varphi_0) \\ L(\varphi_1) \\ \vdots \\ L(\varphi_n) \end{pmatrix}$$

Příklad 1. Najděte aproximaci řešení úlohy

$$\begin{aligned} -0.4y'' + y' &= 2 \\ y(0) &= -1 \\ -0.4y'(1) &= 2y(1) \end{aligned}$$

s krokem $h = \frac{1}{4}$.

Řešení.

$$B(\varphi_0,\varphi_0) = \int_0^{0.25}\left(0.4\cdot(-4)\cdot(-4) + (-4)\cdot(-4x+1)\right)dx = \int_0^{0.25}(6.4+16x-4)\,dx = \int_0^{0.25}(2.4-16x)\,dx =$$

$$= 2.4\,[x]_0^{0.25} - \frac{16}{2}\,[x^2]_0^{0.25} = 2.4\cdot 0.25 - 8\cdot 0.25^2 = \underline{0.1}$$

$$B(\varphi_1,\varphi_1) = \int_0^{0.25}(0.4\cdot 4\cdot 4 + 4\cdot 4x)\,dx + \int_{0.25}^{0.5}\left(0.4\cdot(-4)\cdot(-4) + (-4)\cdot(-4x+2)\right)dx =$$

$$= \int_0^{0.25}(6.4+16x)\,dx + \int_{0.25}^{0.5}(6.4+16x-8)\,dx = 6.4\,[x]_0^{0.25} + \frac{16}{2}\,[x^2]_0^{0.25} - 1.6\,[x]_{0.25}^{0.5} + \frac{16}{2}\,[x^2]_{0.25}^{0.5} =$$

$$= 6.4\cdot 0.25 + 8\cdot 0.25^2 - 1.6\cdot(0.5-0.25) + 8(0.5^2-0.25^2) = \underline{3.2}$$

$$B(\varphi_2,\varphi_2) = \int_{0.25}^{0.5}\left(0.4\cdot 4\cdot 4 + 4\cdot(4x-1)\right)dx + \int_{0.5}^{0.75}\left(0.4\cdot(-4)\cdot(-4) + (-4)\cdot(-4x+3)\right)dx =$$

$$= \int_{0.25}^{0.5}(6.4+16x-4)\,dx + \int_{0.5}^{0.75}(6.4+16x-12)\,dx = 2.4\,[x]_{0.25}^{0.5} + \frac{16}{2}[x^2]_{0.25}^{0.5} - 5.6\,[x]_{0.5}^{0.75} + \frac{16}{2}[x^2]_{0.5}^{0.75} = \underline{3.2}$$

$$B(\varphi_3,\varphi_3) = \int_{0.5}^{0.75}\left(0.4\cdot 4\cdot 4 + 4\cdot(4x-2)\right)dx + \int_{0.75}^{1}\left(0.4\cdot(-4)\cdot(-4) + (-4)\cdot(-4x+4)\right)dx =$$

$$= \int_{0.5}^{0.75}(6.4+16x-8)\,dx + \int_{0.75}^{1}(+6.4+16x-16)\,dx = -1.6\,[x]_{0.5}^{0.75} + \frac{16}{2}[x^2]_{0.5}^{0.75} - 9.6\,[x]_{0.75}^{1} + \frac{16}{2}[x^2]_{0.75}^{1} = \underline{3.2}$$

$$B(\varphi_4,\varphi_4) = \int_{0.75}^{1}\left(0.4\cdot 4\cdot 4 + 4\cdot(4x-3)\right)dx + 2\,\underbrace{y(1)}_{\substack{\varphi_4(1)\\=1}}\,\underbrace{v(1)}_{\substack{\varphi_4(1)\\=1}} = \int_{0.75}^{1}(6.4+16x-12)\,dx + 2\cdot 1\cdot 1 =$$

$$= -5.6\,[x]_{0.75}^{1} + \frac{16}{2}[x^2]_{0.75}^{1} + 2 = \underline{4.1}$$

$$B(\varphi_0,\varphi_1) = \int_0^{0.25}\left(0.4\cdot(-4)\cdot 4 + (-4)\cdot 4x\right)dx = \int_0^{0.25}(-6.4-16x)\,dx = -6.4\,[x]_0^{0.25} - \frac{16}{2}[x^2]_0^{0.25} = \underline{-2.1}$$

$$B(\varphi_1,\varphi_2) = \int_{0.25}^{0.5}\left(0.4\cdot(-4)\cdot 4 + (-4)\cdot(4x-1)\right)dx = \int_{0.25}^{0.5}(-6.4-16x+4)\,dx = -2.4\,[x]_{0.25}^{0.5} - \frac{16}{2}[x^2]_{0.25}^{0.5} = \underline{-2.1}$$

$$B(\varphi_2,\varphi_3) = \int_{0.5}^{0.75}\left(0.4\cdot(-4)\cdot 4 + (-4)\cdot(4x-2)\right)dx = \int_{0.5}^{0.75}(-6.4-16x+8)\,dx = 1.6\,[x]_{0.5}^{0.75} - \frac{16}{2}[x^2]_{0.5}^{0.75} = \underline{-2.1}$$

$$B(\varphi_3,\varphi_4) = \int_{0.75}^{1}\left(0.4\cdot(-4)\cdot 4 + (-4)\cdot(4x-3)\right)dx = \int_{0.75}^{1}(-6.4-16x+12)\,dx = 5.6\,[x]_{0.75}^{1} - \frac{16}{2}[x^2]_{0.75}^{1} = \underline{-2.1}$$

$$B(\varphi_1,\varphi_0) = \int_0^{0.25}\left(0.4\cdot 4\cdot(-4) + 4\cdot(-4x+1)\right)dx = \int_0^{0.25}(-6.4-16x+4)\,dx = -2.4\,[x]_0^{0.25} - \frac{16}{2}[x^2]_0^{0.25} = \underline{-1.1}$$

$$B(\varphi_2,\varphi_1) = \int_{0.25}^{0.5}\left(0.4\cdot 4\cdot(-4) + 4\cdot(-4x+2)\right)dx = \int_{0.25}^{0.5}(-6.4-16x+8)\,dx = 1.6\,[x]_{0.25}^{0.5} - \frac{16}{2}[x^2]_{0.25}^{0.5} = \underline{-1.1}$$

$$B(\varphi_3,\varphi_2) = \int_{0.5}^{0.75}\left(0.4\cdot 4\cdot(-4) + 4\cdot(-4x+3)\right)dx = \int_{0.5}^{0.75}(-6.4-16x+12)\,dx = 5.6\,[x]_{0.5}^{0.75} - \frac{16}{2}[x^2]_{0.5}^{0.75} = \underline{-1.1}$$

$$B(\varphi_4,\varphi_3) = \int_{0.75}^{1}\left(0.4\cdot 4\cdot(-4) + 4\cdot(-4x+4)\right)dx = \int_{0.75}^{1}(-6.4-16x+16)\,dx = 9.6\,[x]_{0.75}^{1} - \frac{16}{2}[x^2]_{0.75}^{1} = \underline{-1.1}$$

Řešme $L(v) = \int_0^1 2v\,dx$

$$L(\varphi_0) = \int_0^{0.25} 2\cdot(-4x+1)\,dx = -\frac{8}{2}\left[x^2\right]_0^{0.25} + 2\left[x\right]_0^{0.25} = \underline{\underline{0.25}}$$

$$L(\varphi_1) = \int_0^{0.25} 2\cdot(4x)\,dx + \int_{0.25}^{0.5} 2\cdot(-4x+2)\,dx = \frac{8}{2}\left[x^2\right]_0^{0.25} - \frac{8}{2}\left[x^2\right]_{0.25}^{0.5} + 4\left[x\right]_{0.25}^{0.5} = \underline{\underline{0.5}}$$

$$L(\varphi_2) = \int_{0.25}^{0.5} 2\cdot(4x-1)\,dx + \int_{0.5}^{0.75} 2\cdot(-4x+3)\,dx = \frac{8}{2}\left[x^2\right]_{0.25}^{0.5} - 2\left[x\right]_{0.25}^{0.5} - \frac{8}{2}\left[x^2\right]_{0.5}^{0.75} + 6\left[x\right]_{0.5}^{0.75} = \underline{\underline{0.5}}$$

$$L(\varphi_3) = \int_{0.5}^{0.75} 2(4x-2)\,dx + \int_{0.75}^{1} 2\cdot(-4x+4)\,dx = \frac{8}{2}\left[x^2\right]_{0.5}^{0.75} - 4\left[x\right]_{0.5}^{0.75} - \frac{8}{2}\left[x^2\right]_{0.75}^{1} + 8\left[x\right]_{0.75}^{1} = \underline{\underline{0.5}}$$

$$L(\varphi_4) = \int_{0.75}^{1} 2\cdot(4x-3)\,dx = \frac{8}{2}\left[x^2\right]_{0.75}^{1} - 6\left[x\right]_{0.75}^{1} = \underline{\underline{0.25}}$$

Maticový zápis soustavy lineárních rovnic:

$$\begin{pmatrix} B(\varphi_0,\varphi_0) & B(\varphi_1,\varphi_0) & 0 & 0 & 0 \\ B(\varphi_0,\varphi_1) & B(\varphi_1,\varphi_1) & B(\varphi_2,\varphi_1) & 0 & 0 \\ 0 & B(\varphi_1,\varphi_2) & B(\varphi_2,\varphi_2) & B(\varphi_3,\varphi_2) & 0 \\ 0 & 0 & B(\varphi_2,\varphi_3) & B(\varphi_3,\varphi_3) & B(\varphi_4,\varphi_3) \\ 0 & 0 & 0 & B(\varphi_3,\varphi_4) & B(\varphi_4,\varphi_4) \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} L(\varphi_0) \\ L(\varphi_1) \\ L(\varphi_2) \\ L(\varphi_3) \\ L(\varphi_4) \end{pmatrix}$$

Nyní uvažujeme okrajovou podmínku $y(0) = -1$ (ze zadání), tj. $y(0) = y_0 = -1 \Rightarrow$ první složku vektoru řešení $y = (y_0, y_1, \dots, y_4)^T$ známe $\Rightarrow$ 1. rovnici můžeme vynechat. Zbývá upravit 2. rovnici systému (rozepíši):

$$B(\varphi_0,\varphi_1)\cdot \underbrace{y_0}_{-1} + B(\varphi_1,\varphi_1)\cdot y_1 + B(\varphi_2,\varphi_1)\cdot y_2 = L(\varphi_1)$$

$\Rightarrow$ převedeme na pravou stranu a dostáváme:

$$B(\varphi_1,\varphi_1)\cdot y_1 + B(\varphi_2,\varphi_1)\cdot y_2 = L(\varphi_1) + B(\varphi_0,\varphi_1) = 0.5 - 2.1 = -1.6$$

Celkem řešíme systém:

$$\begin{pmatrix} B(\varphi_1,\varphi_1) & B(\varphi_2,\varphi_1) & 0 & 0 \\ B(\varphi_1,\varphi_2) & B(\varphi_2,\varphi_2) & B(\varphi_3,\varphi_2) & 0 \\ 0 & B(\varphi_2,\varphi_3) & B(\varphi_3,\varphi_3) & B(\varphi_4,\varphi_3) \\ 0 & 0 & B(\varphi_3,\varphi_4) & B(\varphi_4,\varphi_4) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} \overbrace{L(\varphi_1) + B(\varphi_0,\varphi_1)}^{\text{díky okrajové podmínce}} \\ L(\varphi_2) \\ L(\varphi_3) \\ L(\varphi_4) \end{pmatrix}$$

Po dosazení řešíme systém lineárních rovnic:

$$\begin{pmatrix} 3.2 & -1.1 & 0 & 0 \\ -2.1 & 3.2 & -1.1 & 0 \\ 0 & -2.1 & 3.2 & -1.1 \\ 0 & 0 & -2.1 & 4.1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -1.6 \\ 0.5 \\ 0.5 \\ 0.25 \end{pmatrix}$$

řešení $y = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -0.5673 \\ -0.1956 \\ 0.0593 \\ 0.0913 \end{pmatrix}$ → z okrajové podmínky

Přibližné řešení:

$y_h = -\varphi_0 - 0.5673\,\varphi_1 - 0.1956\,\varphi_2 + 0.0593\,\varphi_3 + 0.0913\,\varphi_4$