# Statistical nonlinear analysis - size effect of concrete beams

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ABSTRACT: This paper presents some results on size effect of concrete beams using statistical nonlinear fracture mechanics analysis. Computer simulation of concrete failure is represented by the program SBETA/ATENA. The fracture of concrete is modeled by the crack band theory based on fracture energy-related softening. Statistical variability of material parameters is introduced in a simple form of small-sample Monte Carlo simulation Latin hypercube sampling. The approach is demonstrated on bending failure of plain concrete beams. Numerical examples are focused on comparison of both deterministic and probabilistic simulation results with experimental data on modulus of rupture.

## 1 INTRODUCTION

The history of description of phenomenon called size effect can be seen as a history of two fundamentally different approaches - deterministic and statistical explanations. First explanation was definitely statistical - it dates back to the pioneering work of Weibull (1939) and many others, mainly mathematicians. Phenomenon that larger specimens will usually fracture under relatively smaller applied load was that time associated with the statistical theory of extreme values. Then most researchers focused on the energetic basis of size effect and the main achievements were purely deterministic, leading to “mean size effect curves”. Let us mention e.g. the book of Bažant & Planas (1998) as an extensive source of information. Uncertainties involved in concrete fracture were considered by some researchers using different theories, from early works e.g. Shinozuka (1972), Mihashi & Izumi (1977), Mazars (1982). Recently, there are attempts to combine last decade´s achievements of both fracture mechanics and reliability engineering e.g. Carmeliet (1994), Carmeliet & Hens (1994), Gutiérez & de Borst (1999) and others.

Recent studies of Bažant & Novák (2000a) on modulus of rupture of plain concrete beams showed the applicability of the probabilistic nonlocal Weibull theory to capture both deterministic and statistical size effects. Those results stimulated the development of statistical-energetic formula by Bažant & Novák (2000b). Extensive test data on the modulus of rupture were used in those papers in order to verify the theory. The authors showed that for small sizes deterministic size effect dominates, but for very large sizes the influence of statistical variability of concrete strength, leading to Weibull type statistical size effect, is decisive.

The aim of this paper is to present some results of the size effect on modulus of rupture obtained by computer simulation of a beam failure using the program SBETA/ATENA. SBETA is a well-known software for nonlinear finite element simulation of concrete and reinforced concrete structures. It employs nonlinear fracture mechanics and energy based concept of concrete fracture (Margoldová et al. 1998, Červenka & Pukl 1999, Červenka 2000). Recently introduced software ATENA is an extension of SBETA. ATENA is conceptually object-oriented and based on a MS-Windows environment (Červenka & Pukl 2000). It includes all material models developed and verified in SBETA. In addition, a wide range of extensions in the ATENA analytical core is available. The user-friendly graphical environment offers excellent support to the user during all stages of the nonlinear analysis. SBETA/ATENA software was used for the deterministic nonlinear analysis of concrete beams of different sizes presented in this paper.

The paper shows the possibility of simple “randomization” of nonlinear analysis of concrete structures. As the scatter of some basic input parameters of computational model (tensile strength of concrete, fracture energy etc.) can be large, it is desirable to consider them to be random variables. Then applying a suitable type of Monte Carlo simulation technique, random variables can be randomly generated under their probability distribution functions. Consequently, the particular problem is repeatedly solved and statistical characteristics of structural response can be obtained and assessed. As nonlinear analysis is computationally intensive, a suitable technique of statistical Monte Carlo simulation should be utilized. The Latin hypercube sampling technique appeared to be a very efficient technique in this context because it requires rather small number of simulations for accurate results (Novák et al. 1998).

Such an approach will be shown on the modulus of rupture size effect problem. The paper presents first selected deterministic simulation results compared to ten sets of experimental data from eight laboratories. Nonlinear fracture mechanics as applied in SBETA/ATENA resulted in a good agreement with these test data (Vořechovský 2000). Second, using a probabilistic approach also the variability of modulus of rupture could be obtained with reasonable computational effort.

The approach, suggested for implementation into SBETA/ATENA package, is general and can be applied for basic statistical analysis of computationally intensive fracture mechanics problems. Advantages and limitations of the approach will also be discussed in the paper.

## 2 NONLINEAR FRACTURE MECHANICS AND SIZE EFFECT

Deterministic size effect represents a transition from ductile behavior of relatively small specimens to brittle behavior of large structures. In the numerical investigations, the ductile behavior can be covered by plasticity, while the brittle behavior corresponds to the linear elastic fracture mechanics (LEFM). For the transition nonlinear fracture mechanics (NLFM) with softening based on fracture energy (Bažant & Planas 1998) can be effectively used. NLFM is suitable for analysis of quasi-brittle materials with certain toughness like concrete. The plastic and the brittle behavior can be treated as limit situations. Due to the energetical basis deterministic size effect can be obtained efficiently by NLFM.

The nonlinear analysis of the concrete beams presented in this paper has been performed by computer program SBETA/ATENA. Suitability of this program for simulation of size effect behavior of concrete structures was reported by Pukl et al. (1992) and by Červenka & Pukl (1993). The constitutive model used in SBETA/ATENA reflects all the essential features of concrete behavior, namely cracking in tension. It is based on nonlinear damage and failure functions in plane stress state (Červenka 2000). A smeared crack approach simulates discrete cracks occurring in real concrete structures by strain localisation in a continuous displacement field. Concrete fracture is covered by nonlinear fracture mechanics based on fracture energy (Margoldová et al. 1998). Exponential softening law derived experimentally by Hordijk (1991) is used. Objectivity of the finite element solution is assured by crack band approach - the descending branch of the stress-strain relationship is adjusted according to the finite element size and mesh orientation.

The transition from ductile (small beam) to brittle (large beam) behavior is documented in Figure 1. The shape of the fracture process zone obtained in SBETA/ATENA simulations for very small (a) and very large (b) beams using data of Sabnis & Mirza (1979) is shown. Corresponding load-deflection diagrams are included.

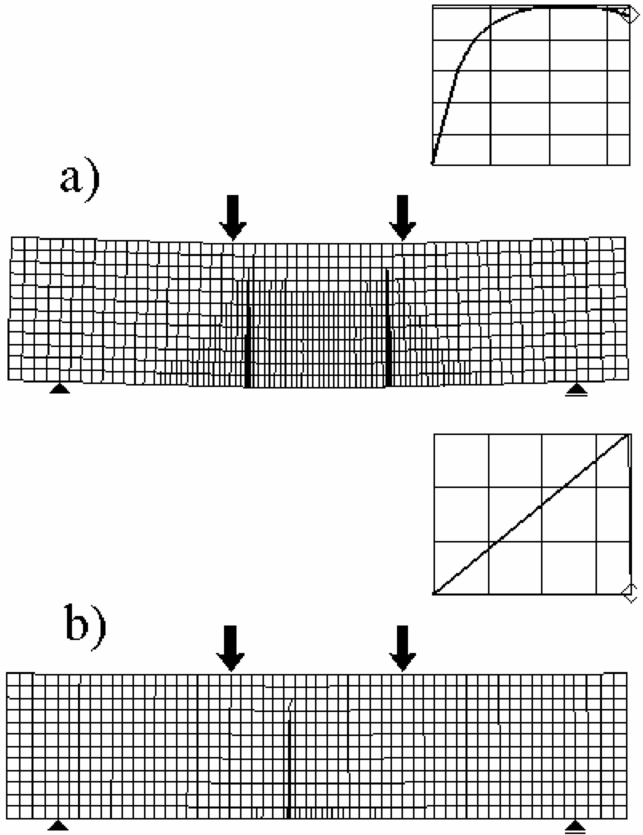


Figure 1. Calculated shapes of fracture process zone and load-deflection diagrams (based on data of Sabnis & Mirza, 1979)
a) small beam (span 0.0381 m),
b) large beam (span 4.0 m).

## 3 SMALL-SAMPLE MONTE CARLO APPROACH: LATIN HYPERCUBE SAMPLING

The classical reliability theory introduced the form of a response variable (deflection, stress, ultimate capacity, crack width etc.) or safety margin (in case that the function expresses failure condition) as the function of basic random variables
$\boldsymbol{X} = X_1, X_2, .., X_i, ..., X_n$

$$R = g(\boldsymbol{X}) \tag{1}$$

where $g(.)$ represents functional relationship between elements of vector $\boldsymbol{X}$ (computational model). Elements of vector $\boldsymbol{X}$ are geometrical and material parameters, load, environmental factors etc., generally uncertainties (random variables or random fields). These quantities can naturally also be statistically correlated.

The primary goal of the statistical analysis is the estimation of basic statistical parameters of response variable $R$, e.g. mean values and variances. Also a histogram and an empirical cumulative probability distribution function are always valuable information. It can easily be done by Monte Carlo simulation, by repetitive calculations of the computational model $g(.)$. But this simple straightforward approach can be time consuming or even impossible to apply as it requires thousands of simulations – repetitive calculations of nonlinear fracture mechanics computational model.

A special type of numerical probabilistic simulation called Latin hypercube sampling (LHS) makes it possible to use only a small number of simulations. This technique originally proposed by McKay et al. (1979) appeared to be a useful reliability technique, different possibilities of efficient use were summarized by Novák et al. (1998). All random variables are divided into $N$ equivalent intervals ($N$ is a number of simulations). This means that the range of the cumulative distribution function $F(X_i)$ of each random variable $X_i$ is divided into $N$ intervals of equal probability $1/N$, (Fig. 2).

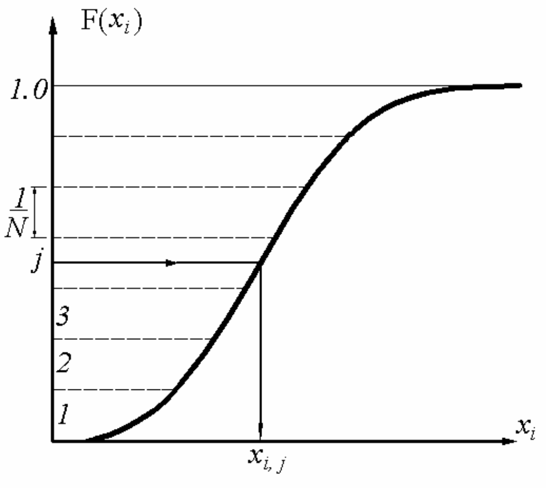


Figure 2. Illustration of LHS – division of cumulative probability distribution function into layers.

The centroids of these intervals are then used in a simulation process. The representative parameters of variables are selected randomly based on random permutations of integers 1, 2, ...,$j$ , ..., $N$. Every interval of each variable is used only once during the simulation. For details see referenced literature. Some improvements of LHS were published recently by Huntington & Lyrintzis (1998). A relatively low number of simulations (say a ten to a hundred) for good estimates of basic statistical parameters proved to be satisfactory. From this point of view this technique is very suitable for complex nonlinear finite element calculations like SBETA/ATENA modeling.

The main reasons for selection of LHS can be summarized as follows:

- EFFICIENCY – good accuracy in statistical characteristics of structural response using a small number of samples (repetitive simulations of nonlinear computationally intensive fracture mechanics calculation).
- SIMPLICITY – the technique is suitable for implementation into complex commercial software like SBETA/ATENA as it requires minor modifications of program core.
- TRANSPARENCY – as it represents an alternative of Monte Carlo simulation the method is transparent and understandable also to people who are not experts in reliability engineering; generally the Monte Carlo type approach is close to engineering thinking.

## 4 PROBABILISTIC MODULE PROPOSED FOR IMPLEMENTATION INTO ATENA

The aim of ATENA statistical nonlinear analysis is to obtain the estimation of the structural response statistics (stresses, deflections, failure load etc.). Procedure can be itemized as follows:

- Uncertainties are modeled as random variables described by theirs probability distribution functions (PDF). The optimal case is if all random parameters are measured and real data exist. Then statistical assessment of this experimental data (e.g. data on strength of concrete or loading) should be done resulting in selection the most appropriate PDF (e.g. Gausian, lognormal, Weibull, etc.). The result of this step is the set of input parameters for ATENA computational model – random variables described by mean value, variance and other statistical parameters (generally by PDF).
- Random input parameters are generated according to their PDF using Monte Carlo LHS type simulation.
- Generated realizations of random parameters are then used as inputs for ATENA computational model. The complex nonlinear solution is performed and results (response) are saved.
- Previous two steps are repeated $N$-times ($N$ is the number of simulations used). At the end of the whole simulation process the resulting set of structural responses is statistically evaluated. The results: Mean value, variance, coefficient of skewness, histograms, empirical cumulative

probability density function of structural response.

The implementation of basic probabilistic methods into ATENA software is now under development. Basically it consists of three main parts:

1 Data input module
 – Selection and description of random variables (theoretical models)
 – Raw measured data input for some variables
 – Statistical correlation (correlation matrix)
 – Definition of response parameter or limit state function for analysis
2 Simulation module
 – Monte Carlo simulation
 – Latin hypercube sampling
 – Simulation monitoring
3 Statistical assessment and basic reliability module
 – Histogram, cumulative probability distribution function, statistical characteristics
 – Statistical testing to obtain the most suitable mathematical model of probability distribution for response variable or safety margin
 – Sensitivity analysis
 – Estimation of failure probability, reliability index, percentiles

## 5 NUMERICAL EXAMPLES – COMPARISON WITH EXPERIMENTAL DATA

### 5.1 *Deterministic comparison with means of test data*

Ten sets of experimental data on modulus of rupture (both three-point and four-point bending) from eight laboratories were used for verification of nonlocal Weibull theory by Bažant & Novák (2000a,b). All details on these data can be found in referenced literature, here only basic material parameters used for the SBETA/ATENA calculation are summarized in Table 1. The results of comparison with means of experimental data are shown as plots of modulus of rupture versus characteristic size $D$ in Figure 3 ($d_a$ is maximum aggregate size). Note, that parameters of high importance are fracture energy and tensile strength of concrete. In most cases these parameters were not tested, therefore estimations had to be accepted. This step was based on:

- Known values of compressive strength and splitting tensile strength;
- Empirical formulas suggested by recommendations of CEB-FIP (1999);
- Statistical prediction of fracture parameters proposed by Bažant & Becq-Giraudon (in press).

In spite of this fact a very good agreement with test data has been achieved in most cases. In Figure 3 also the results of nonlinear fitting of statistical-energetic formula according to Bažant & Novák (2000b) are plotted for comparative purposes. SBETA/ATENA results do not represent a fitting approach that is why larger deviations from test data could occur. The best available experimental data (large amount of information and tested values of material parameters) are data of Rocco (1997), Rokugo (1995) and Sabnis & Mirza (1979). SBETA/ATENA calculations for these data sets are in best agreement.

Table 1. Basic SBETA/ATENA material parameters used for experimental data sets.

| Author | Compressive strength | Direct tensile strength | Modulus of elasticity | Fracture energy |
|---|---|---|---|---|
| | (MPa) | (MPa) | (GPa) | (N/m) |
| Reagel&Willis (1931) | 72 | 4.8 | 50 | 50 |
| Wright, 3point (1952) | 30 | 2.1 | 30 | 25 |
| Wright, 4point (1952) | 30 | 2.0 | 30 | 25 |
| Nielsen (1954) | 47.3 | 2.6 | 40.60 | 20 |
| Lindner&Sprague (1956) | 38.25 | 3.66 | 21 | 70 |
| Walker&Bloem, 1" (1957) | 45 | 4 | 35 | 40 |
| Walker&Bloem, 2" (1957) | 32 | 2.72 | 35 | 120 |
| Sabnis&Mirza (1979) | 36 | 3.85 | 30 | 30 |
| Rokugo (1995) | 40 | 3.20 | 27.5 | 50 |
| Rocco (1997) | 39.1 | 4 | 29.1 | 56.3 |

### 5.2 *Statistical variability of size effect curve*

The presented example represents one of the first attempts to consider random variability of a material in SBETA/ATENA software in a simple form (level of random variables). The best suitable data from the study were selected for that purpose – data of Rocco (1997). For this set also direct tensile strength and fracture energy have been tested. Experimental data show some scatter of modulus of rupture, it is highly probable that it is the consequence of random variability of material properties. Statistical analysis was based on only 8 simulations of LHS – eight realizations of random coefficient normally distributed with unit mean value and standard deviations 0.2 (thus representing variability 20%). The nominal deterministic value of direct tensile strength for Rocco´s data has been multiplied by these realisations – eight direct tensile strengths were considered in eight simulations. Then the most important material parameters of the SBETA/ATENA constitutive model (compressive strength, modulus of elasticity, fracture energy and maximum compressive strain) were obtained from these eight realisations of direct tensile strength utilising recommended formulas of

CEB-FIP (1999). It means that a functional relationship was considered among all these parameters (unit cor- relation coefficient) for simplicity and also because

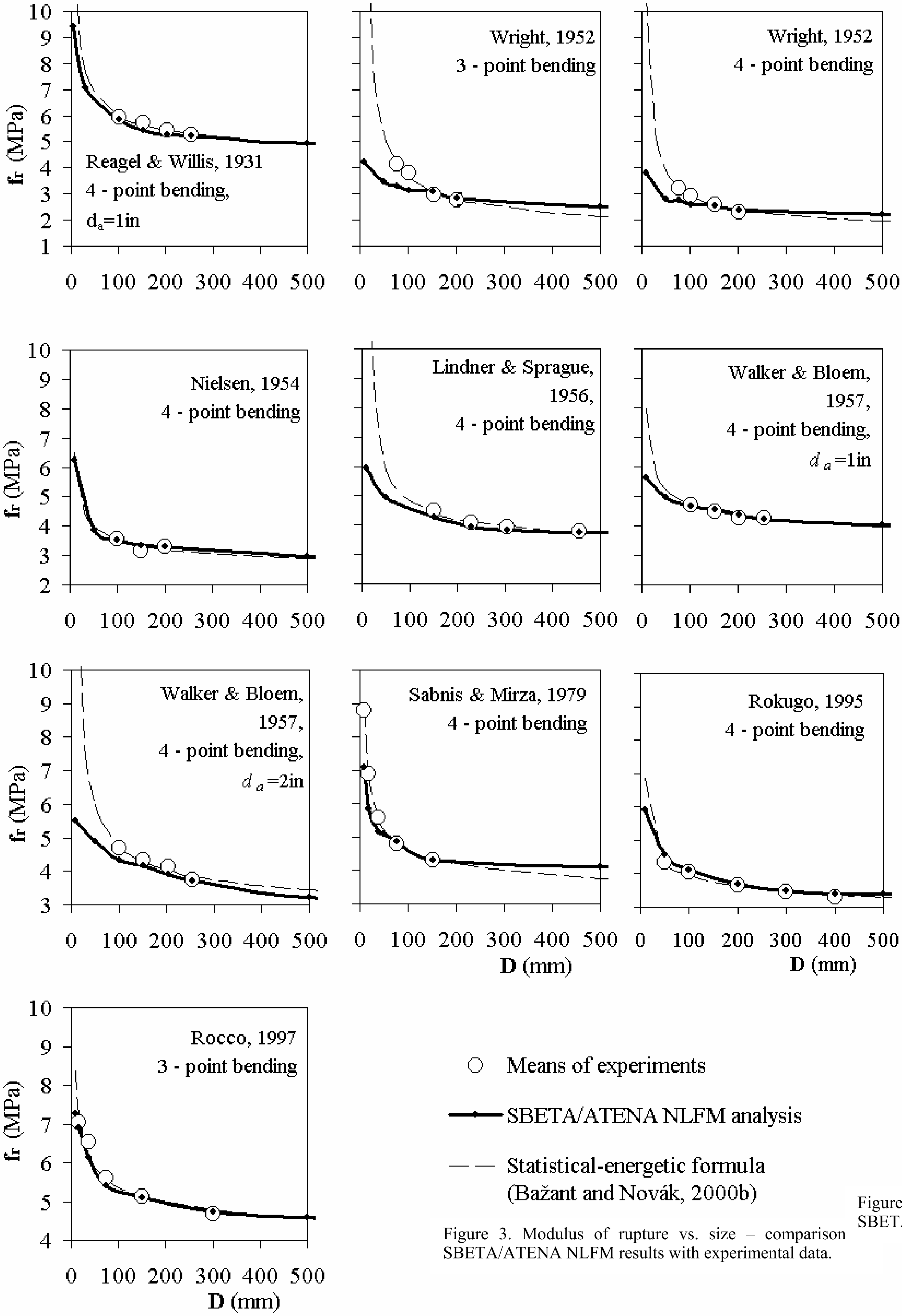


Figure 3. Modulus of rupture vs. size – comparison SBETA/ATENA NLFM results with experimental data.

Figure 3. Modu
SBETA/ATENA

of lack reliable information on this matter.

The analysis resulted in estimation of mean and percentiles size effect curves (modulus of rupture vs. size $D$), as shown in Figure 4. Figure presents mean size effect curve obtained from SBETA/ATENA LHS probabilistic simulation compared to means of test data and percentiles. It can be seen that the scatterband from simulation is larger than in case of tests, which indicates that our assumption of 20% material variability was too rough.

The next important feature can be seen from Figure 4 (and also from Figure 3): Size effect curves have flat horizontal asymptote, mean value (or value from deterministic calculation) of modulus of rupture approaches the direct tensile strength for very large sizes. As it was shown by Bažant & Novák (2000a, b) the slope of this asymptote cannot be zero – a decrease of modulus of rupture with characteristic size continues for very large sizes following the Weibull type size effect. We should emphasize that the aim here was not to capture statistical size effect in the sense of Weibull theory of weakest link concept but to show the influence of randomness of material on the modulus of rupture in a simple form. And naturally the simple form at the level of random variables cannot capture the phenomena of statistical size effect. But the presented approach appeared to capture efficiently the deterministic size effect and its variability.

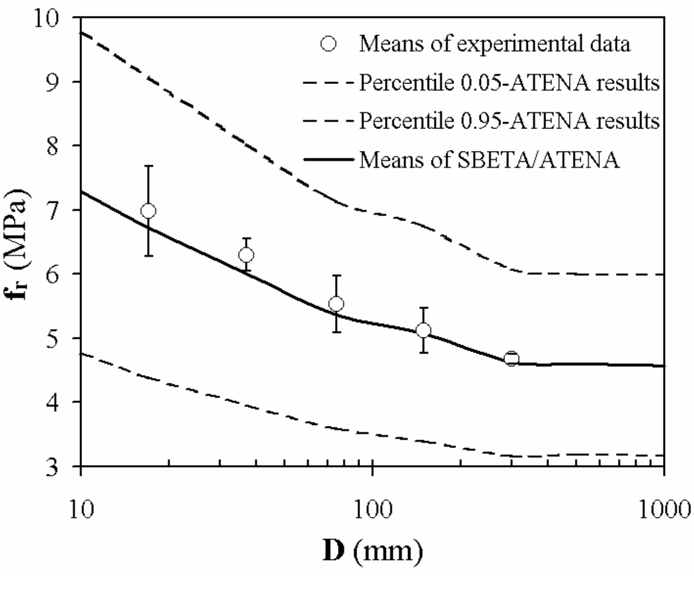


Figure 4. Modulus of rupture vs. size of beam for data of Rocco (1997).

As LHS covers the full range of probabilities, one can estimate the cumulative probability distribution function for the modulus of rupture. This is shown for two sizes (minimum and maximum) of Rocco´s data in Figure 5. As expected, the steepness increases with increasing size, which means that the scatter decreases with size. It agrees with experimental observations, some results of Shinozuka (1972) and detailed numerical verification by Bažant & Novák (2000a).

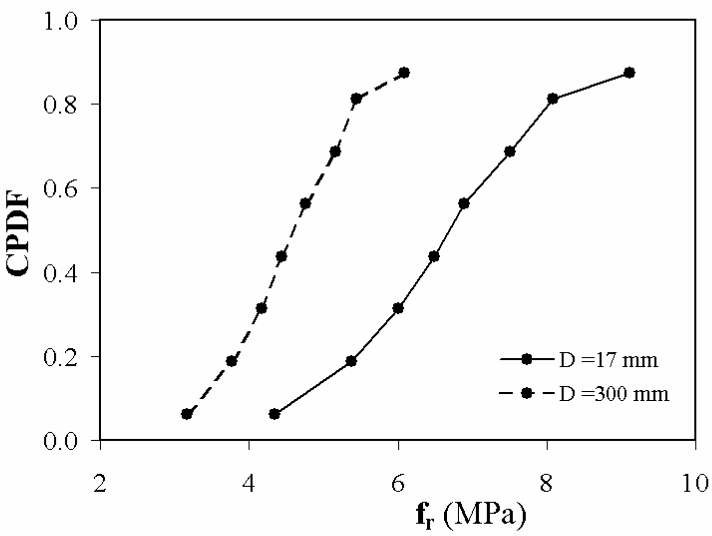


Figure 5. Cumulative probability distribution function of modulus of rupture for extreme sizes of Rocco's data obtained by ATENA statistical calculations.

### 5.3 *Size effect due to bending span of beams in four-point bending*

Koide et al. (1998) tested 279 plain concrete beams in four-point bending, aimed at determining the influence of the beam length $L$ on the flexural strength of beams. These excellent data allow for a comparison of the cumulative probability distribution function (CPDF) of the maximum bending moment $M_{max}$ at failure (Bažant & Novák 2000a). Three sizes of beams with different bending span (200, 400 and 600 mm in series C) are shown in Figure 6. Note that cross-sections of beams were kept constant (0.1m x 0.1m).

Based on experimental data, there is size effect due to bending span as is shown in Figure 7, with increasing bending span the maximum bending moment decreases. Using ATENA deterministic simulation almost a flat size effect curve has been obtained. What is the explanation of this result? The main reason is the difference between the deterministic (energetic) and statistical size effect. Changing only bending span and keeping constant size $D$ resulted in negligible change of energetic deterministic size effect. The decrease of size effect curve in Figure 7 obtained by Koide´s experiment should be attributed to and explained by statistical size effect. As bending span increases there is a higher probability of defect occurence (small strength) in Weibull sense. Therefore the utilization of extreme value statistics in nonlocal Weibull theory by Bažant & Novák (2000a) appeared to be very efficient to verify Koide´s results. Both ATENA deterministic calculation and statistical simulation at the level of random variables cannot capture this phenomenon.

Randomization at the level of random fields (stochastic finite element method) should therefore be the next direction of research in this context.

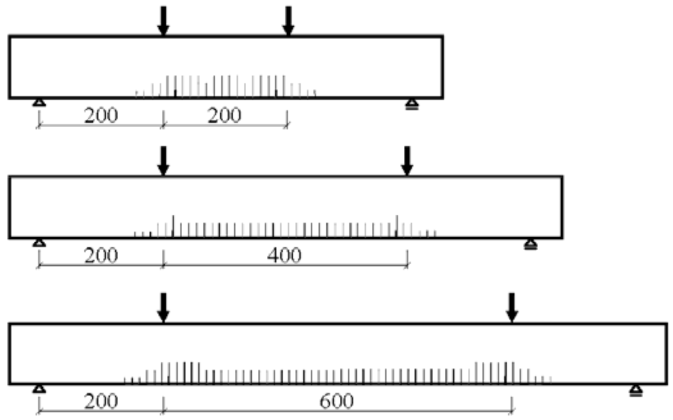


Figure 6. Three sizes of beams (bending span 200, 400 and 600 mm) tested by Koide's (1998), series C.

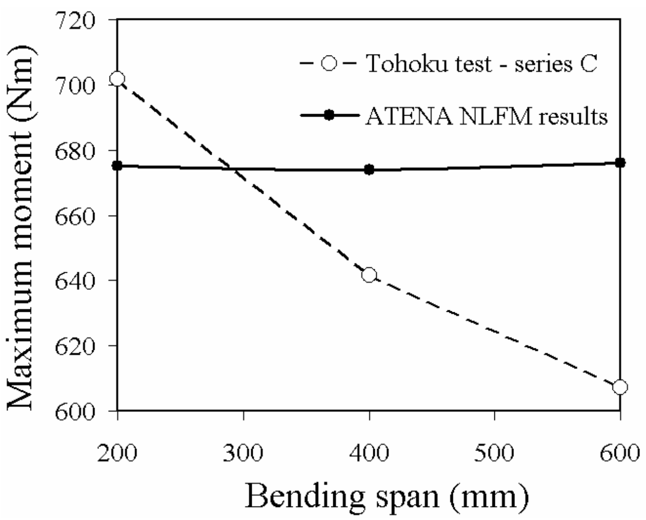


Figure 7. Comparison of means of Koide´s data and ATENA simulations.

## 6 CONCLUSIONS

1 Results of size effect calculations on modulus of rupture obtained by SBETA/ATENA in a deterministic sense are in a very good agreement with experimental data. Generally, it supports the ability of nonlinear fracture mechanics tools implemented into SBETA/ATENA software to capture this phenomenon correctly.
2 The idea of "randomization" of this software has been conceived: It is based on small-sample Monte Carlo approach Latin hypercube sampling which gives accurate estimates of statistical characteristics of response using only a small number of simulations.
3 The feasibility of statistical approach is shown using numerical examples on modulus of rupture. Comparison with statistics of some experimental data is presented: Scatter-band of size effect curve and cumulative probability distribution function of flexural capacity of concrete beams.
4 The results presented here represent outcomes of initial steps of more complex reliability treatment of ATENA software which is under development: statistical, sensitivity and reliability (failure probability calculation) analyses, first implemented at the random variables level and later at the random fields level (stochastic nonlinear finite element method).

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