PROCEDURE OF STATISTICAL SIZE EFFECT PREDICTION FOR CRACK INITIATION PROBLEMS

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ABSTRACT

To simplify stochastic finite element analysis of the strength of structures failing at crack initiation from a smooth surface, the present paper exploits a recently derived [8] generalized law for the combined energetic-probabilistic size effect on the mean nominal strength of structure. The simplification is achieved by using a simple practical procedure recently proposed by the authors [1], which captures both the deterministic and statistical parts of size effect. The paper explains this new procedure and demonstrates its practical application in finite element analysis. In this procedure, Monte-Carlo statistical simulations of nonlinear structural response, which can be very time-consuming, are avoided. It suffices to conduct only deterministic finite element analyses based on nonlinear fracture mechanics, for which the crack band model (or nonlocal model) can be used. The results of deterministic nonlinear finite element analyses for structures of several scaled sizes are then fitted with the deterministic part of the aforementioned size effect law for the mean structural strength. The analysis is simplified, as recently proposed [1], by subdividing the structure into so-called 'random blocks', the strength of which is scaled directly according the stability postulate of extreme value statistics. To capture the statistics, it suffices to superimpose on the fitted deterministic part of size effect law the Weibull size effect, which is important for very large structure sizes. The demonstrated procedure can be performed with any commercial finite element code, provided that the code can satisfactorily reproduce the deterministic size effect.

1 INTRODUCTION AND SIZE EFFECT FORMULAE

To incorporate assessments of the deterministic and statistical size effects into the design or and safety assessment of very large unreinforced concrete structures (such as arch dams, foundations and earth retaining structures), in which the statistical size effect plays a significant role, a simple and robust procedure is necessary. Such a procedure has recently been proposed [1], and in the present paper it is explained and demonstrated. This procedure allows making failure load predictions without simulations of Monte Carlo type. It exploits the previously derived law for the energetic-statistical size effect in the mean sense, and combines this law with deterministic solutions obtained with a finite element code based on nonlinear fracture mechanics, i.e., a code involving the crack band model or nonlocal damage model.

The work of Bažant, Vořechovský and Novák [1], on which this paper is based, utilized an improved size effect law derived in [8], in which the scaling is controlled by two characteristic lengths, one deterministic and one statistical. The objective of this new law is to capture the combined energetic-probabilistic size effect on the nominal strength of structures failing at crack initiation from smooth surface. The role of these two lengths in control the transition from the energetic size effect to the statistical size effect of Weibull type. Their role will here be explained and their relation to the recently developed deterministic-energetic formula and energetic-statistical formula will be clarified.

The deterministic energetic size effect formula for crack initiation from smooth surface [2,3,4], derived as the limit case of equivalent linear elastic fracture mechanics [2,3], reads:

$$\sigma_N\left(D\right)=f_r^{\infty}\left[1+\frac{rD_b}{D+l_p}\right]^{1/r} \tag{1}$$

where σ_N is the nominal strength depending on the structure size, D . Parameters f_r^{∞}, D_b and r are positive constants representing the unknown empirical parameters to be determined. Parameter f_r^{∞} represents the value of the elastic-brittle nominal strength for structure size approaching infinity, which needs to be determined empirically. Exponent r (a constant) controls the curvature and the steepest slope of the size effect curve. This exponent offers a degree of freedom in data fitting while having no effect on the asymptotic expansion of size effect used in the derivation [2,3]. Parameter D_b has the meaning of the thickness of boundary layer of cracking. Variation of parameter D_b moves the entire size effect curve as a rigid body to the left or right in log-log scale; D_b represents a deterministic scaling parameter which is, in principle, related to the grain size (or maximum aggregate size) and drives the transition from elastic-brittle behavior ($D_b/D\rightarrow0$) to quasibrittle behavior ($D_b/D>0$).

In view of the fact that the nominal strength of vanishingly small structures (smaller than D_b) must approach a finite plastic limit, parameter l_p is introduced to control the convergence to this value. Formula (1) represents the full size range transition from perfectly plastic behavior (when $D/D_b\rightarrow0$; $D\ll l_p$) through quasibrittle behavior to elastic brittle behavior ($D/D_b\rightarrow\infty$; $D\gg D_b$). Parameter l_p governs the transition to a small-size asymptotic value corresponding to plastic limit analysis (which is captured in finite element simulations by the crack band model or nonlocal models with spatial averaging of damage). Note that $l_p\neq0$ is required for Eq. (1) to have, for $D\rightarrow0$, a finite limit. This limit is required by the cohesive crack model and was shown to be equal to a solution in which the crack is considered to be filled by a perfectly plastic 'glue'. For large sizes D, the influence of l_p decays fast with increasing D, and so the cases for $l_p\neq0$ and $l_p=0$ are, for large D, indistinguishable.

The large-size asymptote of the deterministic-energetic size effect formula (1) is, in a plot of $\log\sigma_N$ versus $\log D$, horizontal, i.e. $\sigma_N(D)/f_r^{\infty}=1$ (see Fig. 1a). But this is not in agreement with the results for the modulus of rupture obtained with the nonlocal Weibull theory [5], in which the large-size asymptote in the logarithmic plot has the slope $-n/m$ corresponding to the power law of the classical Weibull statistical theory [6] (n is the number of spatial dimensions of scaling, n=1,2,3). Therefore, the statistical and energetic theories need to be superposed. Their superposition is important for large structures, for example, for analyzing the size effect in vertical bending fracture of arch dams, or bending fracture in foundation plinths and retaining walls.

For particular case of glass fibers, the statistical part of size effect and the existence of statistical length scale have been investigated in detail by Vořechovský and Chudoba [7]. Their work shows, briefly, that the large-size (right) asymptote of the statistical part of size effect in structures with stationary random strength field has the classical Weibull form (a straight line of slope $-n/m$ in the plot of $\log\sigma_N$ versus $\log D$), while the small-size (left) asymptote is horizontal. The strength value for the horizontal asymptote for $D/D_b\rightarrow0$ is the mean strength of the random field and, in Weibull sense, it is the mean strength measured for the case when the reference length is equal to the autocorrelation length l_s. So, by introducing the random strength field, one introduces the length scale characterized by l_s. By enriching formula (1) with this statistical result, one gets the final law, derived in [8] by asymptotic reasoning from some relatively plausible hypotheses about nonlocality:

$$\sigma_N=f_r^{\infty}\left[\left(\frac{L_0}{L_0+D}\right)^{rn/m}+\frac{rD_b}{l_p+D}\right]^{1/r} \tag{2}$$

This general law, also presented in [1], exhibits the following salient features:

- By virtue of finite length l_p, this law correctly approaches the small-size (left) deterministic asymptote, corresponding to plastic behavior for vanishingly small sizes.
- Because always $rn/m \ll 1$, the large-size asymptote is the Weibull power law, which represents the purely statistical size effect and corresponds to a straight line of slope $-n/m$ in the double-logarithmic plot of size versus nominal strength.
- The formula involves scaling with two characteristic lengths: deterministic (D_b) and statistical (L_0). It represents the mean size effect as a sum of deterministic and statistical parts, each of which has its own length scale. Parameter D_b controls the transition from quasibrittle to elastic-brittle behavior, and L_0 governs the transition zone between the constant plastic strength and local Weibull strength, via random strength field. The autocorrelation length l_s is related to statistical length L_0, as explained by Vořechovský and Chudoba [7], and by Bažant, Vořechovský and Novák [1].

Including the length in the denominators of Eq. (1) prevents both the statistical and deterministic parts of size effect from growing to infinity for $D/D_b \to 0$. This remedies the problem that the previous energetic-statistical formula [4,5] cannot be applied for arbitrarily small sizes because it intersects the deterministic law at $D=D_b$ and thus gives for very small D a higher mean nominal strength than the deterministic case.

Note that, for $m \to \infty$, Eq. (2) degenerates to deterministic Eq. (1). The same occurs for $L_0 \to \infty$. The interplay of the two characterististic lengths is controlled by the ratio L_0/D_b, as demonstrated in [1]. The question arises as to what is the value of ratio L_0/D_b? Since both characteristic lengths probably are in concrete controlled mainly by the grain size, we expect $L_0 \approx D_b$ and assume $L_0 = D_b$ for practical applications.

2 SUPERPOSITION OF STATISTICAL SIZE EFFECT ON FINITE ELEMENT RESULTS FOR DETERMINISTIC-ENERGETIC SIZE EFFECT

As already mentioned, deterministic modeling with nonlinear finite element programs can capture only deterministic size effect. A procedure of superposing the statistical part of size effect needs to be formulated. According to [1], this can be done as follows:

1) Suppose that the modeled structure has characteristic size (dimension) D_t. The natural first step is to create a finite element computational model for this real size. At this level, one develops the meshing, boundary conditions, material subroutine, etc. Then one obtains a prediction of nominal strength of the structure, corresponding to the peak load on the computed load-deflection diagram, for size D_t. But this reflects only the deterministic-energetic features of fracture, and so the nominal strength of the real structure is usually overestimated at this (first) step, the overestimation being more severe for larger structures. The result of this first step is a point on the size effect plot, represented by the filled circle in Fig. 1a.

2) The second step is to scale down and up the geometry of the computational model, in order to obtain a small set of similar structures with several characteristic sizes D_i, i=1,…, N. Based on numerical experience, N=4 is usually sufficient and more than 10 unnecessary. However, to cover adequately the transition region of size effect, properly chosen sizes D_i must span a broad enough size range, from very small to very large. Then one calculates the nominal strength σ_i, for each size, i=1,…, N. Note that for two very large sizes, the nominal strengths should be almost identical because they must correspond to the horizontal asymptote of the energetic size effect (if not, the failure is not governed by crack initiation alone, and other inelastic phenomena play a non-

negligible role, which means that the present procedure cannot be applied). The computational model must be mesh-insensitive in order to obtain objective results for all the sizes. This means that the crack band model or nonlocal continuum damage model must be used. To ensure that the phenomenon of stress redistribution (causing the deterministic size effect) is correctly captured, well tested models are recommended for the strength prediction. Special attention should be paid to the selection of constitutive law and localization limiter. The result of this step is a set of points (circles) in the size effect plot, as shown in Fig. 1a.

3) The next step is to obtain the optimum fit of the numerical results with the deterministic-energetic formula (1) using the calculated set of N pairs $\{D_i, \sigma_i\}$ i=1,…, t, …, N. The results of this fitting are the values of four parameters: f_r^∞, D_b, r and l_p. Parameter l_p need not be treated as an unknown in fitting, since it can be determined from the fitting based on the plastic analysis (this is explained in [1]). The same applies to parameter f_r^∞ because it can be estimated from the deterministic nonlinear finite element results as the value to which the nominal strength converges at very large sizes. So, one can prescribe (for very large sizes) $\sigma_N / f_r^\infty = 1$ as the asymptotic limit. The outcome of this step is illustrated by the curve fitted to the set of points in Fig. 1a.

4) There are three remaining parameters that need to be ascertained for the statistical-energetic formula (2): n,m and L_0. Parameter n is the number of spatial dimensions (n=1,2 or 3). Parameter m represents the Weibull modulus of the material. A recent study of Bažant and Novák [4] showed that, for concrete and mortar, the correct value of Weibull modulus fitting the asymptotic behavior, is m≈24, rather than 12, the value widely accepted in the past. The ratio n/m represents the slope of the mean size effect curve in the size effect plot for $D/D_b \to \infty$. This means that, for extreme sizes, the nominal strength decreases,

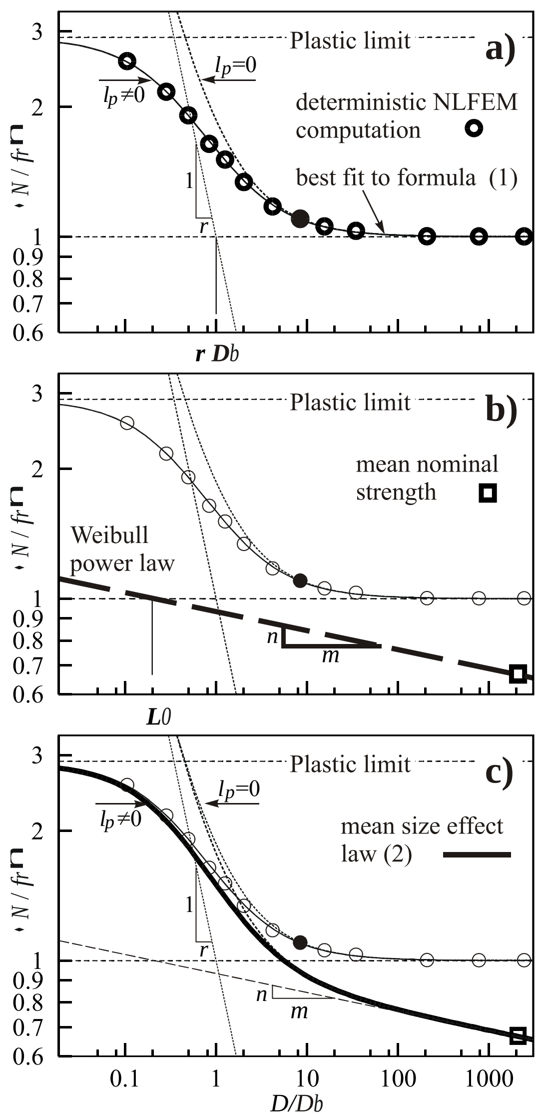

Figure 1: Illustration of the superposition steps. a) Steps 1-4 resulting in deterministic fit; b) step 5 – determination of parameter L_0; c) final formula and nominal strength prediction for the real structural size

for two-dimensional (2D) similarity (n=2), as the −1/12 power of the structure size. Note that, for different materials, the asymptotic values of Weibull modulus are different (e.g., for laminates, often much higher than 24). The results of these 4 steps are shown in Fig. 1a.

Parameter L_0 is now the only remaining parameter to be determined. As it characterizes the statistical length scale, one might think that a statistical software should be incorporated into the finite element code. Not so, however. There is a much simpler alternative, based on a simple evaluation of the local Weibull integral.

The choice of statistical length scale l_s is the primary task (a good choice may be $l_s \approx D_b$). The value of Weibull modulus must be either determined from scatter of small material specimens or assumed from previous experience (m=24), and then one can evaluate by summation from Weibull integral the mean strength of the large size structure considered. This yields the square point in Fig. 1, with coordinates D_{stat} , σ_{stat}. Then one can pass a straight line (Weibull asymptote) of slope $-n/m$ through that point. Graphically, the intersection of the statistical (Weibull) asymptote with the deterministic strength for infinite structure size (horizontal asymptote) f_r^∞ gives the statistical scaling length on the D-axis, see Fig. 1b). The numerical solution to L_0 can be written as:

$$L_0 = D_{stat} \left(\frac{\sigma_{stat}}{f_r^\infty} \right)^{n/m} \tag{3}$$

It is because of this analytical expression that parameter L_0 need not be computed by the fitting of numerical results. For calculating the mean large-size strength σ_{stat} (the square point) from the Weibull integral, one must make a choice of the reference volume V_0 (as well as the Weibull modulus characterizing the scatter); this is discussed in detail, e.g., by Bažant and Planas [3]. The Weibull integral gives the failure probability:

$$P_f = 1 - \exp\left(-\int_V \left\langle \frac{\sigma(\boldsymbol{x})}{s_0} \right\rangle^m \frac{\mathrm{d}V(\boldsymbol{x})}{\boldsymbol{V_0}} \right) \tag{4}$$

where V is the volume (or area, length) of the structure depending on spatial dimension n; s_0 is the Weibull scaling parameter; V_0 is an elementary volume of the material for which the Weibull distribution has parameters m and s_0; and function $\sigma(\boldsymbol{x})$ represents the maximum principal stress at a point of coordinate vector $\boldsymbol{x}$.

One can avoid the computation of this integral (and determination of the mean failure load from P_f) by employing numerical simulation of Monte Carlo type. In such a case, it is recommended to apply the stability postulate of extreme value statistics to scaled random blocks of elements, associating a scaled Weibull pdf with each of these blocks depending on the block size. This effective new approach has been used in the present numerical example and is described in detail by Novák, Bažant and Vořechovský [9,1].

5) After all the parameters of the statistical-energetic formula are determined, the nominal strength of structure can be calculated for any size. Using the real size of the structure, D_t, one can predict the corresponding nominal strength $\sigma_{N,t}$ using Eq. (2). The predicted value will generally be different, and lower, than the initial deterministic prediction (Fig. 1c). The larger the structure, the larger the difference. The formula will provide the prediction for the mean nominal strength. Additionally, a scatter of strength needs to be determined, which can be done easily, just by using the fundamental assumption that Weibull distribution applies. This distribution is fully characterized by two parameters; the Weibull modulus m (or shape parameter), which is prescribed initially,

and the scale parameter *s*, which can be calculated easily from the predicted mean nominal strength and the Weibull modulus.

3 SUMMARY AND CONCLUSIONS

The paper explains a recently derived [8] analytical formula for the mean nominal strength of structures failing at crack initiation, and demonstrates a recently proposed method [1] to exploit this formula for a great simplification of stochastic finite element analysis of structural strength. This new method is simple enough to be used in design practice. It requires only the standard (deterministic) finite element analysis of failure loads of structures of several scaled sizes, and a simple stochastic linear elastic simulation of a structure scaled up to a very large size (which is equivalent to the evaluation of Weibull probability integral from a linear elastic stress field). The entire prediction can be done without complicated and time consuming nonlinear Monte Carlo simulation, which has normally been used to deal with the influence of uncertainties on structural strength.

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