Modeling statistical size effect in concrete by the extreme value theory

Miroslav Vořechovský, PhD Candidate
Institute of structural mechanics, Faculty of Civil Engineering, Brno University of Technology, Brno, Czech Republic

Drahomír Novák, Supervisor

ABSTRACT: The paper analyzes various modeling alternatives for the statistical size effect in quasibrittle structures. The problematic feature of stochastic finite element method: how to capture the statistical part of the size effect phenomenon for structures of very large structures is tackled. The idea is to emulate the recursive stability property from which the Weibull extreme value distribution is derived. The feasibility of proposed approach is demonstrated on the crack initiation problem and predictions are compared with the existing test data.

Keywords: Size effect, extreme value theory, random strength, fracture energy, chain of bundles

1 INTRODUCTION

Large concrete structures usually fracture under a lower nominal stress than geometrically similar small structures (the nominal stress being defined as the load divided by the characteristic cross section area). This phenomenon, called the size effect, has in general two physical sources - deterministic and statistical. The deterministic source consists of the stress redistribution and the associated energy release described by nonlinear fracture mechanics (in finite element setting, the crack band model or cohesive crack model). The deterministic size effect represents a transition from ductile failure with no size effect, asymptotically approached for very small structures, to brittle failure with the strongest possible size effect, asymptotically approached for very large structures [2].

The classical explanation of size effect used to be purely statistical - simply the fact that the minimum random local strength of the material encountered in a structure decreases with an increasing volume of the structure. Although what became known as the Weibull distribution was in mathematics discovered already in 1928 by Fisher & Tippett [5] (in connection with Tippett's studies of the length effect on the strength of long fibers).The need for this extreme value distribution in describing fatigue fracture of metals and the size effect in structural engineering was first developed, independently of Fisher & Tippett's cardinal contribution, by Weibull (1939) [16]. His pioneering work was subsequently refined by many other researchers, mainly mathematicians; e.g. [4,14].

The stochastic finite element method (SFEM) should be able to capture both types of the size effect phenomenon if the deterministic FEM solver accounts for nonlinearities correctly. The decisive parameter in SFEM is the correlation length which governs spatial correlation over the structure. The correlation length modifies the size effect curve in the region where this parameter is smaller than the element size. There is a clear relationship - the larger the correlation length, the stronger is the spatial correlation of strength along the structure and, consequently, the weaker is the decrease (due to local strength randomness) of the nominal strength with increasing structure size. Computational problems, however, develop in trying to simulate the extreme value asymptotic size effect using the random field approach. Approximately, the requirement is that the ratio of the correlation length to the element size should not drop bellow one. This poses a major obstacle to using SFEM for describing the size effect, especially for large structure sizes.

Some advances in this problem were achieved by several authors, e.g. [7] who, however, confined their studies to the size range of real structures. The ratio of the correlation length to the element size implies, unfortunately, a severe limitation. To actually compute the extreme value asymptote using the random field approach, the number of discretization points (e.g. nodes in a finite element mesh) would have to increase proportionally to the structure size, which is in practice impossible since an extremely large structure size would have to be considered to approach the asymptotic behavior closely. To make computations feasible, it is necessary to devise a way to increase the element size in proportion to the structure size, keeping the number of elements constant.

Therefore, the aims of this paper are: (i) To introduce the problem by summarizing the vital features of the statistics of extremes established by mathematical statisticians in a form meaningful to engineers, putting emphasis on the philosophy of derivation of the probability distribution of extreme values in a set of independent stochastic variables having an arbitrary elemental probability distribution, (ii) To draw the consequence for capturing the statistical size effect with the help of SFEM, (iii) To propose a method for computer simulation of the statistical size effect based directly on the basic concept of extreme value statistics in combination with nonlinear fracture mechanics, and verify it by an example.

2 WEAKEST LINK CONCEPT AND THEORY OF EXTREME VALUES

The weakest link concept for the strength of a chain-like structure with *N* elements is equivalent to the distribution of the smallest values in samples of size *N*. If one element, the weakest element, fails, the whole structure fails, i.e., the failure is governed solely by the element of the smallest strength. To clarify the problem, it will be useful to recall some basic formulae of the statistical theory of extremely small values. The strength distribution of an element of a chain-like (or statically determinate) structure, i.e., the distribution of the failure probability of an element as a function of the applied stress σ, may be characterized by continuous probability density function $p_1(\sigma)$ with the associated cumulative distribution function $P_1(\sigma)$ (in statistical literature called the elemental, underlying, basic or primary distribution). Then the cumulative distribution of the failure probability of a structure of *N* elements (or the distribution of the smallest strength value in samples of size *N*) is given by

$$P_N(\sigma) = \int_{-\infty}^{\sigma} p_1(\sigma)\,\mathrm{d}\sigma = 1 - \left(1 - P_1(\sigma)\right)^N \tag{1}$$

and the failure probability density reads

$$p_N(\sigma) = np_1(\sigma)\left[1 - P_1(\sigma)\right]^{N-1} \tag{2}$$

These basic equations provide an overall representation of the failure distribution $p_N(\sigma)$ (or $P_N(\sigma)$) corresponding to a given elemental distribution $p_1(\sigma)$. Different elemental distributions can give different failure distributions $P_N(\sigma)$, however, it is remarkable that the asymptotic forms $P_N(\sigma)$ can be only three.

3 IMPLICATIONS FOR THE FINITE ELEMENT METHOD

The strength distribution of such a structure is known, based on a mathematical argument. Therefore, one needs to consider the large size asymptotic behavior and verify that it conforms to this distribution. The asymptotic behavior rests on the so-called stability postulate of extreme value statistics, generally accepted beginning with Fréchet (1927) [6]. In this postulate, the extreme value of a set of $\nu = Nn$ identical independent random variables *X* (the strengths) is regarded as the extreme of the set of *n* extremes of the subsets of *N* variables. When both $n \to \infty$ and $N \to \infty$, it is perfectly reasonable to postulate that the distribution of the extreme of set *Nn* must be similar to the distribution of the extreme of each subset *N* (i.e., related to it by a linear transformation). In other words, the asymptotic form of the distribution must be *stable*. From this property it can be shown that the survival probability f_N of a structural system with a very large size N as a function of applied strength σ must asymptotically satisfy the functional equation

$$\left[f(\sigma)\right]^N = f\left(a_N\sigma + b_N\right) \tag{3}$$

where a_N and b_N are functions of size N. In the most important paper of extreme value statistics motivated by the strength of textile fibers, Fisher & Tippett [5] showed that this recursive functional relation for function $f(\sigma)$ can be satisfied by three and only three distributions. One of them had already been found by Fréchet [6] and the other two have later become known as the Gumbel and Weibull distributions (curiously, not the Fisher and Tippett distributions). The first two distributions have no threshold and admit negative values of the argument, and so are unsuitable for strength. Hence, the Weibull distribution is the only realistic distribution for structural strength.

Consequently, the only way to ensure the correctness of SFEM for failure analysis is to make it match the large size asymptotic behavior, in particular, the Weibull power law size effect, typical of structures failing at crack initiation. But how to overcome the obstacle of a forbiddingly large number of random variables associated with all the finite elements?

The basic idea proposed here is to exploit directly the fundamental stability postulate from which Fisher & Tippett [5] derived the asymptotic forms of the extreme value distributions. In regard to SFEM, this postulate may be literally implemented as follows: Instead of subdividing a very large structure into the impracticably large number ν of finite elements having the fixed size of the characteristic volume, we must use a mesh with only n macroelements (or finite elements) associated with n random strength variables, keeping n fixed and increasing the macroelement size with the structure size, while the subdivision N of each macroelement is defined as the ratio of its volume to the characteristic volume of the material. Then each of these n subsets of N variables is simulated statistically, and for each subset the extreme is selected to be the representative statistical property of the finite element (macroelement). These n extremes of the subsets of N variables are then used in FEM analysis of the whole structure. This procedure ensures that the extreme value statistics is correctly approached, with one crucial advantage - the number n of finite elements (macroelements) remains reasonable from the computational point of view. Although N increases with the structure size, the determination of the extreme from the subdivision of each macroelement does not add to the computational burden since it is carried out outside FEM analysis.

One basic hypothesis of the classical Weibull theory of structural strength is the statistical independence of the strengths of the individual characteristic volumes l_0^2 (in 2D) or l_0^3 (in 3D), where l_0 is the characteristic length. The strength of each of these volumes can be described by Weibull distribution with Weibull modulus m and scale parameter σ_0 (threshold being taken as zero, as usual for strength). Each of the aforementioned macroelements, whose characteristic size is L_0 and characteristic volume L_0^2 or L_0^3, may be imagined of being discretized into N characteristic volumes l_0^2 or l_0^3, i.e. $N = L_0^2 / l_0^2$ or $N = L_0^3 / l_0^3$. This consideration provides the statistical properties of the macroelement. Since we are interested only in very small tail probabilities, we may substitute in these equations the tail approximation of the elemental (generic) Weibull distribution with a certain modulus and scale parameter. The tail approximation is the power function σ^m (times a constant), and its substitution leads for the strength of the macroelement again to Weibull distribution but with a different modulus and scale parameter, and thus with a different mean and variance, which are expressed as follows (COV is kept constant):

$$\mu = \mu_0\left(N\right)^{-1/m} = \sigma_0\Gamma\left(1+1/m\right)\left(N\right)^{-1/m} \tag{3}$$

$$\sigma^2 = \mu\left(\frac{\Gamma\left(1+2/m\right)}{\Gamma^2\left(1+1/m\right)} - 1\right) \tag{4}$$

4 NUMERICAL EXAMPLE: SPAN SIZE EFFECT IN FOUR-POINT BEND BEAM TESTS

4.1 *Experiment and attempt at deterministic simulation*

Abundant experimental evidence on the statistical size effect on plain concrete beams has been accumulated by now in the literature. Recently, Koide et al. [8,9] tested 279 plain concrete beams under four-point bending, aimed at determining the influence of the beam length L on the flexural strength of beams. These excellent data permit a comparison of the cumulative probability distribution function (CPDF) of the maximum bending moment $M_{\max}$ at failure [1,12]. Beams of three different bending spans, 200, 400 and 600 mm (series C of Koide et al. [8,9]) are shown in Figure 1, along with the cracks obtained by deterministic finite element calculations (with the code ATENA [3]). The cross-sections of all the beams were kept constant (0.1m times 0.1m). The experimental data show that $M_{\max}$ decreases as the span increases. To explain this size effect of the span, shown in Fig. 3, Koide et al. provided a Weibull theory based approach.

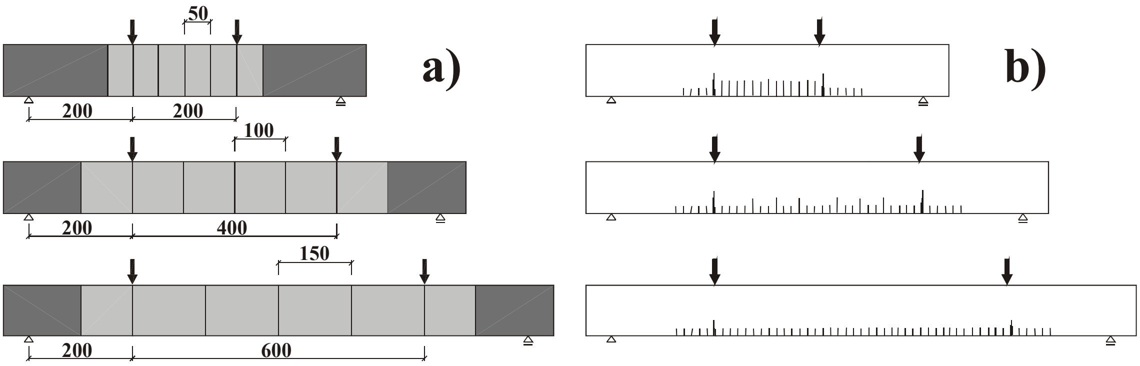

Figure 1. a) Koide's beams of bending span 200, 400 and 600 mm, series C. b) Crack patterns for deterministic calculations for the three sizes.

Unfortunately, only the compression strength of the concrete used is known, whereas the direct tensile strength and fracture energy have not been tested. The experimental data depicted in Figure 3 represent the mean values for each size. The double-logarithmic plot of $M_{\max}$ versus the span forms a straight line with a slope $D^{-\dim/m}$, where dim is the spatial dimension, m is the Weibull modulus and D is the bending span. The problem is properly analyzed as one-dimensional, and then the overall slope of the experimental data in the figure is matched best using m=8 (which is an unusually low value for concrete, indicating a relatively high scatter).

Deterministic simulation with nonlinear fracture mechanics software ATENA yields results consistent with a flat size effect curve, i.e., absence of size effect. This is not surprising. According to fracture mechanics, there is almost no deterministic size effect in flexure of unreinforced beams when the beam depth is not varied because the energy release function is almost independent of the beam span. This is useful for our focus on the statistical size effect. It allows a purely statistical analysis of the test data in Figure 3, reflecting the fact that, the longer the beam, the higher is the probability of encountering in it a material element of a given low strength.

In finite element simulations, the beams were loaded by force increments in order to avoid a nonsymmetric bending moment distribution when the crack pattern (Fig. 2) becomes nonsymmetric, due to material randomness. The load-deflection curves, including the peak and post-peak, were calculated under load control using the arc length method.

4.2 *Statistical size effect*

The probabilistic version of nonlinear fracture mechanics software ATENA [3] was utilized to simulate the tests of Koide et al. by finite elements, in accordance with the theory of extreme values. This was made possible by integrating ATENA with the probabilistic software FREET [10,13,15].

In this simulation, the finite element mesh is defined by using only 6 stochastic macroelements placed in the central region of test beams in which fracture initiates randomly; see Fig. 2. The chosen macroelements have the form of strips. The strips suffice for simulating the Weibull size effect. We imagine N elements per macroelement of width L_0, while the finite element meshes for all the sizes are identical (except for a horizontal stretch).

The characteristic length is considered to be approximately 3-times the maximum aggregate size, i.e., about 50 mm. The Weibull modulus is taken as m=8, and the scale parameter is 1.0. The statistical parameters of the strength of the macroelements, imagined to consist of $N = L_0/l_0$ independent elemental material elements each, are calculated from (3). For the three sizes (spans) considered here, L_0 = 50, 100, 150 mm and N = 1, 2, 3.

In the present approach, a stochastic computational model with 6 random tensile strength variables is defined for each beam size (span); 16 random simulations of these 6 statistically independent variables, based on the method of Latin hypercube sampling, are performed using FREET and ATENA software [10,11,13,15]. The statistical characteristics of the ultimate force can then be evaluated. The mean values of nominal strength obtained from a statistical set of maximum forces are determined first. Figure 3 shows the random cracking pattern at failure, obtained for four realizations of three progressively improved alternatives of solution.

To illustrate the random failures, the corresponding random load-deflection curves are shown in Figure 3. The three alternatives, for which the results are presented in Fig. 3, are as follows:

4.3 *Alternative I:*

The first alternative is a pure Weibull type approach in which only the random scatter of tensile strength is considered, the generic mean value of tensile strength being fixed as 3.7 MPa. For the three sizes (spans) considered here then, according to formulas (3) and (4) the means of tensile strengths are μ = 3.484, 3.195 and 3.037 MPa, coefficient of variation COV = 0.148 (driven by the Weibull modulus m only).

The resulting size effect curve obtained by probabilistic simulation is found to have a smaller slope than the experimental data trend, in spite of the fact that an unusually low Weibull modulus (m=8) is used. This can be explained easily. The Weibull theory strictly applies only when the failure occurs at crack initiation, before any (macroscopically) significant stress redistribution with energy release. However, the material, concrete, is relatively coarse, the test beams not being large enough compared to the aggregate size, and so a non-negligible fracture process zone must form before a macroscopic crack can form and propagate, dissipating the required fracture energy G_f per unit crack surface. Therefore, the beam, analyzed by nonlinear fracture mechanics (the crack band model, approximating the cohesive crack model) does not fail when the first element fails (as required by the weakest link model imitating the failure of a chain). Rather, it fails only after a group of elements fails, and several groups of failing elements can develop before the beam fails; see Figure 2. The finite element simulations are able to capture this behavior thanks to the cohesive nature of softening in a crack, reflecting the energy release requirement of fracture mechanics.

4.4 *Alternative II*

The idea to overcome the problem and match the size effect data is to take the randomness of fracture energy G_f into account (Weibull distribution). Using the generic mean of fracture energy, G_f = 93 N/m, for the three spans, according to formulas (3) and (4) the means of fracture energy are G_f = 87.6, 80.3 and 76.3 N/m, COV 0.148. The generic mean of tensile strength is again μ = 3.7 MPa. But we cannot ignore the statistical correlation of G_f to tensile strength. For lack of available data, we simply assume a very strong correlation, characterized by correlation coefficient 0.9. Such a correlation tends to cause an earlier onset of (macroscopic) crack propagation, compared to Alternative I. The result is shown in Figure 3 as Alternative II. The resulting slope of the simulated size effect curve is now close to the slope of experimental data. However, the whole curve is shifted down, i.e., all the beams are weaker than they should be. It can be seen that the strong correlation between tensile strength and fracture energy causes the macroelements with a lower tensile strength to be more brittle. The failure, therefore, localizes into these macroelements (Fig. 2).

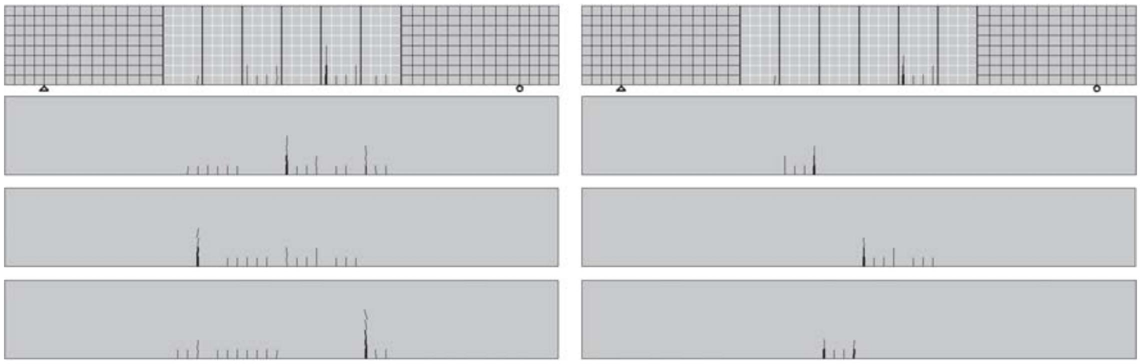

Figure 2. Macroelements (random variables) and examples of random crack initiation for the smallest beam size; left: random tensile strength only (Alternative I); right: random and highly correlated tensile strength with fracture energy.

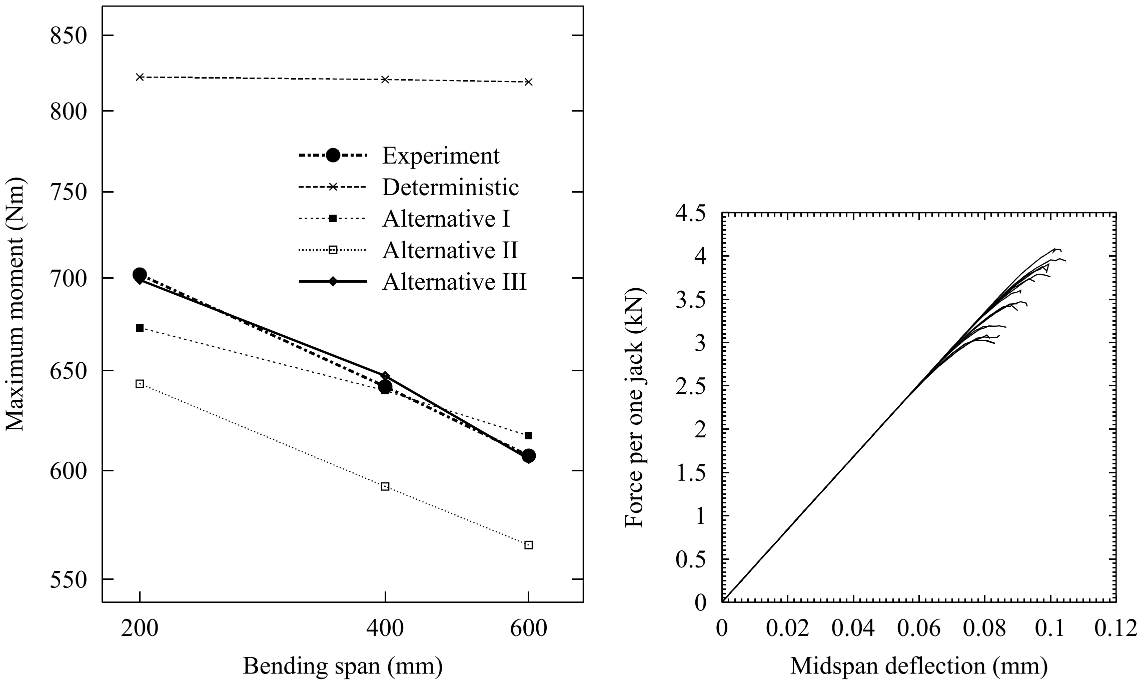

Figure 3. Left: Comparison of Koide's data and the deterministic and statistical simulations. Right: Random load-deflection curves.

4.5 *Alternative III*

In seeking a remedy, we must realize that Koide et al. have not measured the tensile strength nor the fracture energy, and our foregoing estimate may have been too low. So a heuristic approach is the only option. While keeping *Aternative II*, we are free to shift the size effect curve up by increasing the generic mean value of tensile strength and the fracture energy value. We increase them to 4.1 MPa and 102 N/m, respectively, and this adjustment is found to furnish satisfactory results; see Figure 3. Although the size effect of *Alternative III* in the double logarithmic plot is not as straight as the trend of data, the differences from the data are negligible. These small differences may have been easily caused, for instance, by insufficient size of the calculated data set, or by weaker numerical stability near the peak load, making a precise detection of the peak (under load control) less accurate.

5 DISCUSSION

The third alternative leading to a good mean size effect curve (MSEC) still does not mean that the statistical size effect is captured correctly. In order to ensure this one would have to check the whole distribution function for each size. If the real structure is brittle or if it fails right after the first crack initiation, the distribution function for each size must be Weibull and moreover, the COV (or *m*) must be the same, constant and equal to those used for the elemental strength distribution. Although Koide measured the strength for a relatively high number of specimens for each size (35-40) this still does not allow us to draw conclusions about the whole distribution, mainly the tail behavior.

Although we approached more "brittle" behavior by high positive correlation between the strength and fracture energy, the load-deflection diagrams (Fig. 3) still shows considerable strength redistribution before the peak is reached. This may cause deviation from the simple 1-dimensional Weibull statistical model illustrated on Fig. 4.

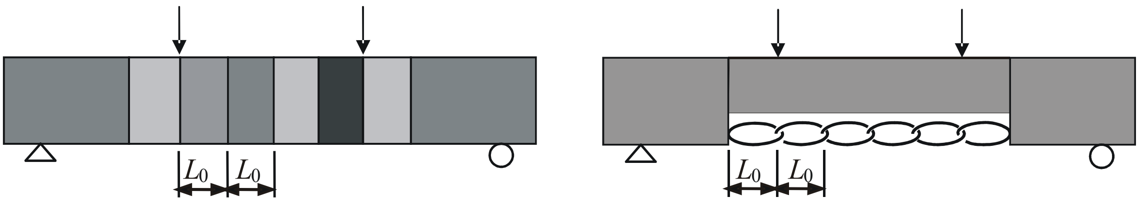

Figure 4. Simple 1-D Weibull model (chain). Left: Random variables – macroelements. Right: a statistical "chain" model driving the strength of a whole structure.

In case of quasibrittle materials, such as concrete, the largest beams must fail after development of relatively short cracks (in comparison with the beam depth *D*). But this is not the case for small beams. When the peak load is reached the crack length is not negligible and this calls for more sophisticated statistical model illustrated on the Figure 5.

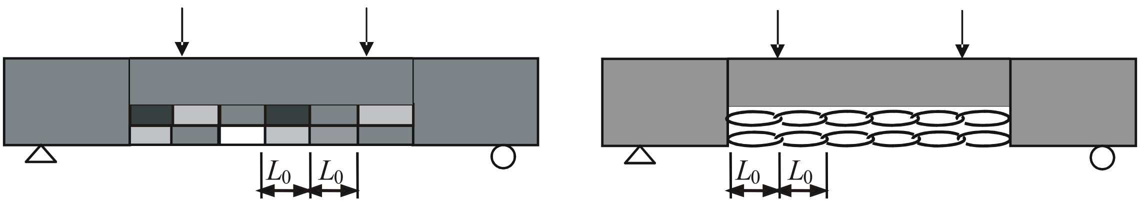

Figure 5. More sophisticated statistical model for small sizes - the bundle of chains model.

Since the deterministic size effect is highly suppressed by having the depth *D* constant, we may expect the MSEC curve inclined due to statistical size effect only (straight line in double-logarithmic plot). However, the influence of different load-sharing rules in case of the model in Fig. 5 may cause deviation of statistics for all sizes from the pure Weibull type scaling. This is under investigation now.

6 CONCLUSIONS

The problematic feature of stochastic finite element method (SFEM): How to capture the statistical size effect for structures of very large sizes is tackled. A simple and effective strategy for capturing the statistical size effect using SFEM is developed. The idea to emulate the recursive stability property from which the Weibull extreme value distribution works well for large enough structures where the deterministic part of size effect (causing the stress redistribution) is weak. Using Monte Carlo simulation and computational modeling of nonlinear fracture mechanics, a probabilistic treatment of complex fracture mechanics problems is rendered possible. The approach is demonstrated by simulating the size effect in plain concrete beams under four-point bending, for which extensive statistical test data have recently been reported by Koide et al.

7 ACKNOWLEDGMENTS

The work of was supported by the Grant Agency of the Czech Republic under the grant number 103/02/1030 and under project reg. No. CEZ: J22/98: 261100007.

REFERENCES

[1] Bažant, Z. P. & Novák, D. 2000. Probabilistic Nonlocal Theory for Quasibrittle Fracture Initiation and Size Effect. II: Application, *Journal of Engineering Mechanics*, Vol. 126, No. 2: 175-185.
[2] Bažant, Z. P. & Planas, J. 1998. *Fracture and Size Effect in Concrete and Other Quasibrittle Materials CRC Press*, Florida, and London: Boca Raton, (chapter 12).
[3] Červenka, V. & Pukl, R. 2002. *ATENA - Computer Program for Nonlinear Finite Element Analysis of Reinforced Concrete Structures*. Program documentation. Czech Republic: Červenka Consulting.
[4] Epstein, B. 1948. Statistical Aspects of Fracture Problems. *Journal of Applied Physics*, Vol. 19: 140-147.
[5] Fisher, R. A. & Tippett L. H. C., 1928. Limiting Forms of the Frequency Distribution of the Largest and Smallest Member of a Sample, Proc., *Cambridge Philosophical Society*, Vol. 24: 180-190.
[6] Fréchet, M. 1927. Sur la loi de probabilité de l' écart maximum. *Ann. Soc. Polon. Math*. 6, 93.
[7] Gutiérrez, M.A. & de Borst, R. 2001. Deterministic and probabilistic length-scales and their role in size-effect phenomena. In R.B. Corotis et al, (eds.), ICOSSAR 2001 (Proc., 8th Int. Conf. on Structural Safety and Reliability, held in California, USA), Swets & Zeitinger, 1-8.
[8] Koide, H., Akita, H. & Tomon, M. 1998. Size Effect on Flexural Resistance due to Bending Span of Concrete Beams, In H. Mihashi and K. Rokugo, (eds.), Fracture Mechanics of Concrete Structures (Proc., 3rd Int. Conf., FraMCoS-3, held in Gifu, Japan), Freiburg, Germany: Aedificatio Publishers.
[9] Koide, H., Akita, H., & Tomon, M. 2000. Probability Model of Flexural Resistance on Different Lengths of Concrete Beams. In R. E. Melchers and M. G. Stewart, (eds.), Application of Statistics and Probability (Proc., 8th Int. Conf., ICASP-8, held in Sydney, Australia 1999), Rotterdam: Balkema.
[10] Novák, D. et al. 2002. *FREET - Feasible Reliability Engineering Efficient Tool*. Program documentation, Brno University of Technology, Faculty of Civil Eng. / Červenka Consulting, Czech Republic.
[11] Novák, D., Bažant. & Vořechovský, M. 2003. Computational modeling of size effect in quasibrittle structures. In Kiureghian (ed.) Proc., ICASP 9, San Francisco, USA.
[12] Novák, D., Vořechovský M., Pukl, R. & Červenka, V. 2001. Statistical nonlinear fracture analysis: Size effect of concrete beams. In de Borst et al. (eds.) Int. Conf. On Fracture Mechanics of Concrete Structures FraMCos 4, Cachan, France. Rotterdam: Balkema.
[13] Novák, D., Vořechovský, M. & Rusina, R. 2003. Small-sample probabilistic assessment - FREET software . In Kiureghian (ed.) Proc., ICASP 9, San Francisco, USA.
[14] Saibel, E. 1969. Size Effect in Fracture, In Proc. Int. Conf. on Structure, Solid Mechanics and Engineering Design in Civil Engineering Materials, Southampton, part 1, 125-130.
[15] Vořechovský, M. & Novák, D. 2003. Small-sample probabilistic assessment - FREET software . In Kiureghian (ed.) Proc., ICASP 9, San Francisco, USA.
[16] Weibull, W. 1939. The Phenomenon of Rupture in Solids. In Proc., Royal Swedish Institute of Engineering Research, Stockholm: Ingenioersvetenskaps Akad. Handl., Vol. 153, 1-55.

Miroslav Vořechovský, PhD Candidate
Brno University of Technology
Faculty of Civil Engineering
Institute of Structural Mechanics
Veveří 95
662 37 – Brno – Czech Republic
Tel. : + 420 5 4114 7370
Fax : + 420 5 4124 0994
E-mail: vorechovsky.m@fce.vutbr.cz

Prof. Ing. D. Novák, DrSc., Supervisor
Brno University of Technology, E-mail: novak.d@fce.vutbr.cz