Probabilistic Determination of the Number of Fibers Bridging a Crack in Short Fiber Reinforced Composites

Miroslav Vořechovský and Václav Sadílek

***Abstract*—This paper derives the probabilistic distribution functions for a random number of fibers that intersect a plane in a composite specimen reinforced by short fibers. Fibers placed and oriented randomly in 3D space bridge matrix cracks with a certain inclination angle and embedded length distributions. The distributions are determined for two alternatives: (A) the fibers' centers have completely homogeneous distribution inside the specimen volume and (B) the fibers have increased density in the bulk so that no fiber can protrude from the specimen boundaries. Additionally, we derive the joint probability density function of a random embedded length and inclination angle to the normal of the (crack) plane for those fibers that intersect that plane.**

***Index Terms*—short-fiber reinforced composites, conditional probability and statistics.**

I. Introduction

SHORT fiber reinforced concrete (FRC) is a relatively new material that extends concrete's sphere of application. Today, FRC is routinely used e.g. for industrial floors. A common requirement on such composite materials is a ductile response in later loading stages which announces the total failure of a structural element. The fibers positively modify the macroscopic behavior of the composite and increase its compressive strength, tensile strength, stiffness, fracture toughness, impact resistance, etc. This behavior can be achieved by adding a certain proportion of short fibers to the matrix mixture, mostly in an amount between 1–3 vol%. Experience has shown that such an amount of fibers effectively bridge cracks in the matrix and force it to develop many distributed cracks rather than a few widely opened cracks. Therefore, a large amount of energy can be dissipated prior to reaching the peak load or deformation. Such fibers are made of steel, alkali-resistant glass, aramid, carbon or various other materials.

In order to correctly determine the (random) force carried by fibers that bridge a crack plane, one has to ascertain (i) how many fibers really bridge the crack plane and (ii) the inclination and position of the fibers with respect to the crack plane. This is the focus of the present paper. We derive the aforementioned characteristics based on (a) natural assumptions regarding the distribution of the random positions and orientations of fibers inside the volume of the FRC specimen, and (b) the available information about the shape of the composite specimen and the shapes of the fibers, and also the volume fraction of the fibers.

M. Vořechovský and V. Sadílek are with the Department of Structural Mechanics, Faculty of Civil Engineering, Brno University of Technology, Veveří 95, 602 00 Brno, Czech Republic, e-mail: vorechovsky.m@fce.vutbr.cz, sadilek.v@fce.vutbr.cz.

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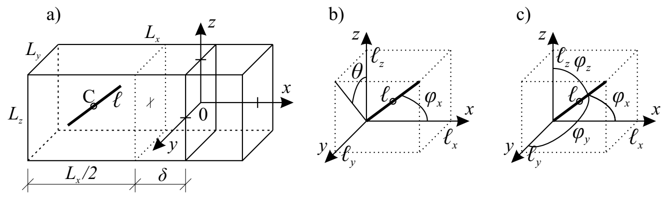

Fig. 1. Fiber geometry – a) Placement of the coordinate system inside specimen volume $L_x L_y L_z$. b) Definition of the fiber orientation for computer generation: $\ell_x = \ell\cos(\varphi_x)$, $\ell_y = \ell\sin(\theta)\sin(\varphi_x)$, $\ell_z = \ell\cos(\theta)\sin(\varphi_x)$. c) Definition of fiber orientation (angles): $\ell_x = \ell\cos(\varphi_x)$, $\ell_y = \ell\cos(\varphi_y)$, $\ell_z = \ell\cos(\varphi_z)$.

The assumptions in the present work are: (i) possible fiber collisions in the specimen volume are ignored due to the small volume fraction of the fibers, and therefore the fibers are considered independent of each other; (ii) the fiber orientation is statistically homogeneous in the specimen volume, i.e., we are not considering e.g. the dependence of fiber orientation on concrete flow direction as described in [1] and [2]; (iii) we ignore the possibility of the spatial dependence of the local density of fibers due to e.g. the occurrence of clusters of fibers developed during the fabrication process, etc.

We remark that in former studies, homogenization approaches were developed [3], [4], after deriving the distribution functions for random orientation and position, respectively, of fibers bridging a discrete planar crack. These approaches, however, do not take into account the variability in crack bridge performance which is caused by both the scatter in the response of a single fiber and the scatter in the total number of bridging fibers. Furthermore, the statistical dependency between the distribution of the position and the inclination angle of short fibers was omitted in these works. The randomness of these quantities introduces variability to the crack bridge response and has a significant influence on the reliability of composites with just a single crack; it has an even more pronounced influence on the strength reduction if multiple cracking occurs.

II. Geometry of randomly distributed fibers

The position and orientation of each fiber in a three-dimensional FRC specimen can be uniquely described by five variables. Suppose the composite is a solid block of the

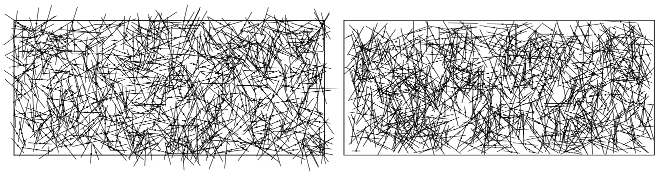

Fig. 2. FRC specimen generated by Monte Carlo – view in the y direction. Left – fibers can protrude from the specimen body (alternative (A)), right – all fibers inside the specimen body (alternative (B)).

volume $V_{\mathrm{t}} = L_x L_y L_z$ (see Fig 1a). We assume that all three dimensions (lengths) exceed twice the length of the fiber ℓ. We place the origin of the (aligned) coordinate system into the specimen centroid and then we shift the specimen in x direction by a constant δ while keeping the coordinate system unchanged (the shift δ fulfills $|\delta| \leq L_x/2$). Any fiber position can be defined by the three Cartesian coordinates of the fiber centroid (midpoint) x, y and z, and its orientation can be defined by a pair of independent angles, see Fig. 1. From all possible pairs of angles, we select (i) the angle φ_x between the fiber and x-axis, and (ii) the angle φ_y between the fiber and y-axis, see Fig. 1c. The fibers are supposed to be randomly placed and oriented within the specimen region. Therefore, we treat the five geometrical descriptors as random variables.

This paper considers two different alternative fiber distributions: (A) the fibers' centers have completely homogeneous distribution inside the specimen volume (the specimen boundaries have no effect, see Fig. 2 left), and (B) the fibers have modified position densities depending on their orientation in the vicinity of the specimen boundaries (all fibers must be wholly inside the specimen, see Fig. 2 right). For both alternatives it is assumed that the distribution of the angular orientations is independent of the fibers' center coordinates x, y and z.

In order to find the distribution functions of the two angles that lead to the equal representation of all possible fiber orientations, we have to take the density of the angle φ_x to be

$$f_{\varphi_x}(\varphi_x) = \begin{cases} \sin\varphi_x & \text{for } \varphi_x \in \langle 0; \frac{\pi}{2} \rangle \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and the density of the angle φ_y to be

$$f_{\varphi_y}(\varphi_y) = \begin{cases} \frac{1}{2}\sin\varphi_y & \text{for } \varphi_y \in \langle 0; \pi \rangle \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Note that for computer generation, it is convenient to work with angle θ uniformly distributed over the interval $(0, 2\pi)$ instead of angle φ_y, see Fig 1b.

Based on the densities of the two independent angles, we now construct the joint probability density functions (PDFs) of the positions and orientations of the fibers. In both studied alternatives, the fiber center position is assumed to be uniformly distributed; the difference is that the uniform distribution is over different intervals.

A. Fiber position independent of orientation

In alternative (A), the joint probability density function of all five independent variables defining the fiber position and orientation is just the product of five marginal densities:

$$f^A_{xyz\varphi_x\varphi_y}(x, y, z, \varphi_x, \varphi_y) = \sin(\varphi_x)\frac{\sin(\varphi_y)}{2}\frac{1}{L_x L_y L_z} \tag{3}$$

From the joint distribution it is easy to obtain e.g. the density of variable x (random fiber position w.r.t. to the x-axis). This distribution is uniform over interval S^A:

$$S^A_x = \left\langle -\frac{L_x}{2} + \delta; +\frac{L_x}{2} + \delta \right\rangle \tag{4}$$

The other two coordinates have the intervals:

$$S^A_i = \left\langle -\frac{L_i}{2}; +\frac{L_i}{2} \right\rangle \tag{5}$$

where i is either y or z. The marginal probability density of x can be written with the help of the interval (4) as:

$$f_x(x) = \begin{cases} \dfrac{1}{L_x} & \text{for } x \in S^A_x \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

B. Fiber position dependent on orientation

In alternative (B), the fiber placement is considered uniformly distributed over lengths $L^\star_x, L^\star_y$ and $L^\star_z$, which are dictated by fiber orientations φ_x and φ_y:

$$\begin{aligned} L^\star_x &= L_x - \ell\cos(\varphi_x) \\ L^\star_y &= L_y - \ell\left|\cos(\varphi_y)\right| \\ L^\star_z &= L_z - \ell\sqrt{1 - (\cos(\varphi_x))^2 - (\cos(\varphi_y))^2} \end{aligned} \tag{7}$$

Therefore, the joint PDF reads:

$$f^B_{xyz\varphi_x\varphi_y}(x, y, z, \varphi_x, \varphi_y) = \sin(\varphi_x)\frac{\sin(\varphi_y)}{2}\frac{1}{L^\star_x}\frac{1}{L^\star_y}\frac{1}{L^\star_z} \tag{8}$$

In this case, the probability density function of variable x (random fiber position w.r.t. to the x-axis) is the conditional probability density function conditioned by the angle φ_x:

$$f_{x|\varphi_x}(x|\varphi_x) = \frac{f_{x\varphi_x}(x, \varphi_x)}{f_{\varphi_x}(\varphi_x)} = \begin{cases} \frac{1}{L^\star_x} & \text{for } x \in S^B_x \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

which is the uniform distribution over the interval:

$$S^B_x = \left\langle -\frac{L_x}{2} + \frac{\ell}{2}\cos\varphi_i + \delta; +\frac{L_x}{2} - \frac{\ell}{2}\cos\varphi_x + \delta \right\rangle \tag{10}$$

The other two fiber centers have the uniform distribution over the intervals

$$S^B_i = \left\langle -\frac{L_i}{2} + \frac{\ell}{2}\cos\varphi_i; +\frac{L_i}{2} - \frac{\ell}{2}\cos\varphi_i \right\rangle \tag{11}$$

where i is taken either as y or z.

The five-dimensional domain of the identified random variables is specified in (1), (2), (4) and (10). One can easily check that in both alternatives, the integral of the joint PDFs over the domains of all five variables is equal to one.

The above-specified joint density functions (3) and (8) should be used in general 3D considerations. From here on, the focus is mainly on the situation in which the specimen is cut by a plane perpendicular to the x-axis (see Fig. 1a) and the fibers intersected by this plane are considered. In this particular application, the positions y and z within the plane of the intersected fibers and their orientation φ_y do not influence the solutions and are therefore irrelevant. In other words, we disregard the coordinates y and z and the orientation angle φ_y from here on. The fiber placement can be fully described by the joint density function of the pair x and φ_x. The corresponding joint distributions can be obtained from (3) and (8) by integrating over all possible y, z and φ_y. In alternative (A), the joint probability density function reads:

$$f^A_{x\varphi_x}(x,\varphi_x) = \frac{\sin(\varphi_x)}{L_x} \tag{12}$$

while in alternative (B) it equals:

$$f^B_{x\varphi_x}(x,\varphi_x) = \frac{\sin(\varphi_x)}{L_x - \ell\cos(\varphi_x)} \tag{13}$$

III. Probability of a fiber intersecting a plane

In order to calculate the force carried by fibers bridging a crack (assumed to be planar) one has to determine the total number of fibers intersecting the crack. This number, k, is a random variable. The task of calculating the distribution of the number of intersecting fibers k out of the total number of fibers n inside the composite can be simplified to evaluating the probability of a single fiber being intersected by a (crack) plane. Once this probability, p, is known for a certain position of the plane, one can use the binomial distribution for the number of intersecting fibers k within the composite. The probability of a fiber intersecting a plane is obtained as a portion of the total unit probability of all possible fiber placements. This portion is calculated as an integral over all possible configurations with a suitable indicator function that signals the presence of a fiber within the intersecting plane. The fiber length projection can be written as $\ell\cos(\varphi_x)$, and the condition for the fiber to intersect the plane reads:

$$-\frac{1}{2}\,\ell\cos\varphi_x \le x \le \frac{1}{2}\,\ell\cos\varphi_x \tag{14}$$

The events of a plane intersecting or missing a fiber can be conveniently written using the indicator function $\mathbf{1}_A(x)$. The indicator function indicates the membership of an element in a subset A of X, having the value 1 for all elements of A (intersection) and the value 0 for all elements of X not in A. Mathematically, the following indicator function can be used:

$$\mathbf{1}_A(x) = \mathbf{1}_A\left(\frac{1}{2}\,\ell\cos(\varphi_x) - |x|\right) \tag{15}$$

We note that the fiber is assumed to be very thin and therefore the effect of its diameter is ignored in the indicator function. The crack is assumed not to occur in the immediate vicinity of the boundary zones (defined by the distance of one fiber length ℓ).

From here on we denote $L \equiv L_x$ and $\varphi \equiv \varphi_x$.

A. Fiber position independent of orientation

In alternative (A), the probability that the fiber intersects a crack plane is calculated as an integral of the joint density function (12) multiplied by the cutting condition (indicator function) over the domain of all possible positions x and orientations φ:

$$p^A = \int_0^{\frac{\pi}{2}} \int_{-\frac{L}{2}+\delta}^{\frac{L}{2}+\delta} \frac{\sin\varphi}{L}\,\mathbf{1}_A\left(\frac{\ell}{2}\cos\varphi - |x|\right)\mathrm{d}x\mathrm{d}\varphi \tag{16}$$

When the cutting plane is half-length of the fiber away from the boundaries, i.e. within interval I: $|\delta| < \frac{1}{2}(L-\ell)$, the probability of cutting a fiber is independent of the cut position – the solution to (16) is simply:

$$p^A_{\mathrm{I}}(\delta;\ell;L) = p(\ell;L) = \frac{1}{2}\frac{\ell}{L} \tag{17}$$

Otherwise, for the intervals II close to the boundaries, i.e. when $|\delta| \ge \frac{1}{2}(L-\ell)$, the probability decreases to

$$\begin{aligned} p^A_{\mathrm{II}}(\delta;\ell;L) = \frac{1}{4\ell L}&\left[\ell^2 - 3L^2 - 4\delta\ell + 2L\ell\right.\\ &\left.+\,12L\delta - 12\delta^2\right] + \frac{(L-2\delta)^2}{2\ell L} \end{aligned} \tag{18}$$

At the limit (when $\delta = \frac{L}{2}$), the probability drops to $p = \frac{1}{4}\frac{\ell}{L}$.

B. Fiber position dependent on orientation

In alternative (B), the probability of intersection equals

$$p^B = \int_0^{\frac{\pi}{2}} \int_{-\frac{L}{2}+\delta+\frac{\ell}{2}\cos\varphi}^{\frac{L}{2}+\delta-\frac{\ell}{2}\cos\varphi} \frac{\sin\varphi}{L-\ell\cos\varphi}\,\mathbf{1}_A(x)\,\mathrm{d}x\mathrm{d}\varphi \tag{19}$$

If the specimen length is twice as large as the fiber length ℓ, and the distance of the cutting plane from the specimen edge is larger than one fiber length, i.e. $|\delta| < \frac{L}{2} - \ell$, then the interval I is defined as either:

$$|\delta| \le \frac{1}{2}(L-\ell) \quad \vee \quad |\delta| \ge \frac{1}{2}(L+\ell) \tag{20}$$

The probability (19) in interval I reads:

$$p^B_{\mathrm{I}}(\delta;\ell;L) = \frac{L}{\ell}\ln\left(\frac{L}{L-\ell}\right) - 1 \tag{21}$$

It can be checked that as L becomes larger, p^B_{I} tends to p^A_{I}. Otherwise, in intervals II, i.e. in the vicinity of boundaries:

$$\delta \ge \ell - \frac{L}{2} \quad \wedge \quad \delta \ge \frac{L}{2} - \ell \tag{22}$$

the probability of intersection reads

$$\begin{aligned} p^B_{\mathrm{II}}(\delta;\ell;L) = \frac{1}{2\ell}&\left[(L+2\delta)\ln\left(\frac{2}{L+2\delta}\right)\right.\\ &\left.-\,(\ln(L-\ell)+1)(L-2\delta) + 2L\ln(L)\right] \end{aligned} \tag{23}$$

IV. NUMBER OF FIBERS INTERSECTING A PLANE

We will now discover how many fibers, out of the total number n inside the specimen, are intersected by a (crack) plane (see Fig. 3). The total number of fibers n inside the composite can be calculated from the fiber volume fraction v_f, the volume of a single fiber V_f, and the total specimen volume V_t:

$$v_\mathrm{f} = \frac{nV_\mathrm{f}}{V_\mathrm{t}} = \frac{nA_\mathrm{f}\ell}{A_\mathrm{c}L} \quad \Rightarrow \quad n = v_\mathrm{f}\frac{A_\mathrm{c}L}{A_\mathrm{f}\ell} \tag{24}$$

where A_f is the cross-sectional area of a single fiber and A_c is the specimen's cross-sectional area L_yL_z.

The process of cutting a specimen containing n fibers can be modeled as n independent Bernoulli trials each with a probability of success equal to p. The probability p of intersecting a fiber has been derived in the previous section and it takes a value from the interval $(0, 1)$. The random variable k has the binomial distribution $\mathrm{Bi}(n, p)$. Therefore, the probability mass function for the random number k reads:

$$p_k = \mathrm{P}(K = k) = \binom{n}{k} p^k (1-p)^{(n-k)} \tag{25}$$

The mean value and variance of the number of fibers intersecting a crack plane reads:

$$\begin{aligned} E\left[k\right] &= \mu_k = np \\ D\left[k\right] &= \sigma_k^2 = np\left(1-p\right) \end{aligned} \tag{26}$$

This binomial distribution can be approximated by the Poisson distribution with the parameter $\lambda = np$. Asymptotically, the distribution of k can also be approximated by the Gaussian distribution with the mean value and variance given by (26); see Fig. 4 for a comparison of the distribution functions.

V. GEOMETRY OF FIBERS INTERSECTED BY A PLANE

In this section, we study the distribution of (i) random angle and (ii) embedded length for fibers that intersect a (crack) plane. We remark that the results are obtained only for planes (cracks) placed away from the boundary zones, i.e. within interval I. We define two random variables: (i) angle φ_c of a fiber that intersects the plane and (ii) embedded length ℓ_e, which is the *shorter length* of a bridging fiber found on either the left or the right side of the crack plane.

The probability density function of angle φ_c can be obtained by integrating the joint density function (12) or (13), multiplied by the indicator function and reduced by the probability p_I, over the domain of x (i.e. either S_x^A or S_x^B):

$$f_{\varphi_\mathrm{c}}(\varphi) = \frac{1}{p_\mathrm{I}} \int\limits_{S_x} f_{x\varphi}\left(x,\varphi\right) \mathbf{1}_A \left(\frac{\ell}{2}\cos\varphi - |x|\right) \mathrm{d}x \tag{27}$$

In alternative (A), the probability density of angle φ_c^A under which the fibers intersect a (crack) plane reads:

$$f_{\varphi_\mathrm{c}}^A\left(\varphi_\mathrm{c}\right) = \begin{cases} \sin\left(2\varphi_\mathrm{c}\right) & \text{for } \varphi_\mathrm{c} \in \left\langle 0; \frac{\pi}{2} \right\rangle \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

Thus, the mean value (=median=modulus) of φ_c

$$E\left[\varphi_\mathrm{c}\right] = \int\limits_{\varphi_\mathrm{c}=0}^{\frac{\pi}{2}} \varphi_\mathrm{c} \sin\left(2\varphi_\mathrm{c}\right) \mathrm{d}\varphi_\mathrm{c} = \frac{\pi}{4} \tag{29}$$

In other words, the most frequent angle of a fiber intersecting the crack plane is 45°. This is important when considering the snubbing effect of fibers – the most frequent bridging fiber is considerably inclined.

In alternative (B), the density of angle φ_c^B under which the fibers intersect a (crack) plane reads:

$$f_{\varphi_\mathrm{c}}^B\left(\varphi_\mathrm{c}\right) = \begin{cases} \frac{\ell\sin(2\varphi_\mathrm{c})}{2(L-\ell\cos\varphi_c)p_\mathrm{I}^B} & \text{for } \varphi_\mathrm{c} \in \left\langle 0; \frac{\pi}{2} \right\rangle \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

The embedded length of the fiber ℓ_e is, given the fiber's center position x and orientation φ_x, calculated as:

$$\ell_e = \max\left(0\ ,\ \frac{\ell}{2} - \frac{|x|}{\cos\varphi_x}\right) \tag{31}$$

The embedded length ℓ_e in the central part (I, both alternatives) of the specimen length is uniformly distributed over the interval $(0, \frac{\ell}{2})$ with the PDF:

$$f_{\ell_e}\left(\ell_e\right) = \frac{2}{\ell} \tag{32}$$

and the mean value equals

$$E\left[\ell_e\right] = \frac{1}{4}\ell \tag{33}$$

It can be shown that in the central part of the specimen (interval I), the joint probability density function of the pair (φ_c and ℓ_e) can be constructed simply as a product of the marginal densities, i.e. products of (28) and (32) in alternative (A), and the product of (30) and (32) in alternative (B).

VI. NUMERICAL EXAMPLE

In order to check the above analytical results, Monte Carlo numerical simulations were performed. Let us consider a specimen with the dimensions $L_x = 0.1$ m, $L_y = L_z = 0.04$ m, and the length of steel fibers $\ell = 0.01$ m, fiber diameter $d_\mathrm{f} = 0.3$ mm, number of fibers $n = 1\,000$ (this corresponds to the volume fraction $v_\mathrm{f} = 0.44$ %), and the cutting plane placed in the center of the specimen $\delta = 0$. Figure 4 shows the results of alternative (B), i.e. all fibers are inside the specimen volume. We performed 10 000 Monte Carlo simulations and the obtained empirical histogram is very well approximated by the distributions mentioned in section IV (Binomial, Poisson and Normal). The mean value and standard deviation of the number of fibers intersecting the plane for all distributions is approximately $\mu \approx 53.4$ and $\sigma \approx 7.1$. In alternative (A), the figure is similar and the mean value is smaller ($\mu \approx 50$), and the standard deviation $\sigma \approx 7.0$.

VII. CONCLUSION

The paper studies the random number of fibers that bridge a crack plane in a short fiber reinforced composite. By exploiting the information on the volume fraction of fibers, and of the shape both of the specimen and of the fiber, it is shown how to calculate the random number of fibers bridging a crack plane for two alternatives: (A) the fibers' centers have a completely homogeneous distribution inside the specimen volume and (B)

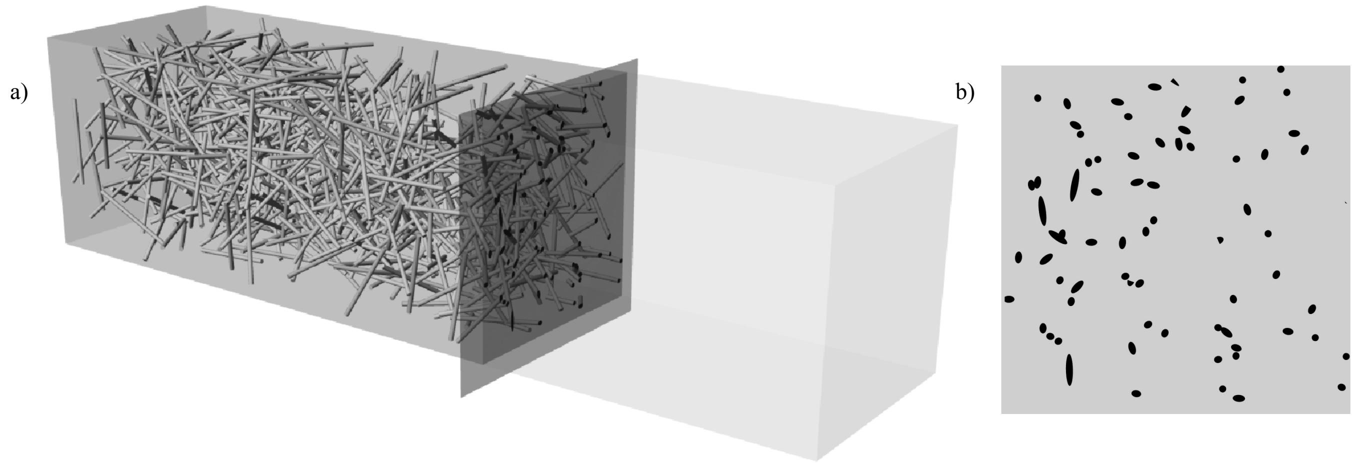

Fig. 3. a) Fibers randomly distributed inside a 3D volume and cut by a plane. b) View of the cutting plane (crack).

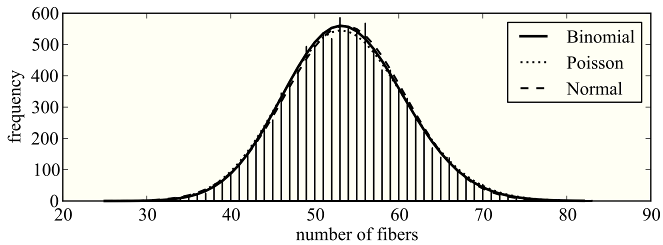

Fig. 4. Histogram of the random number of fibers intersecting the crack plane compared to Binomial, Poisson and Normal distributions.

the fibers have increased density in the bulk so that no fiber can protrude from the specimen boundaries.

The derived distributions are important for the correct determination of the random force carried by fibers that bridge a crack plane in randomly reinforced composites. This is shown in [5], where the derived distributions are utilized to probabilistically evaluate the crack bridging force by computing the sum of a random number (of bridging fibers) of independent random variables (single fiber responses, including their pullout behavior).

REFERENCES

[1] P. Stähli, R. Custer, and J. G. M. Mier, "On flow properties, fibre distribution, fibre orientation and flexural behaviour of FRC," *Materials and Structures*, vol. 41, no. 1, pp. 189–196, 2007.

[2] K. Zhou and J. Lin, "Three dimensional fiber orientation distribution in two dimensional flows," *Fibers and Polymers*, vol. 9, no. 1, pp. 39–47, 2008.

[3] V. C. Li, Y. Wang, and S. Backer, "A micromechanical model of tension-softening and bridging toughening of short random fiber reinforced brittle matrix composites," *Journal of the Mechanics and Physics of Solids*, vol. 39, no. 5, pp. 607–625, 1991.

[4] Z. Lin, T. Kanda, and V. C. Li, "On interface property characterization and performance of fiber-reinforced cementitious composites," *Journal of Concrete Science and Engineering*, vol. 1, pp. 173–184, 1999.

[5] M. Vořechovský, V. Sadílek, and R. Rypl, "Probabilistic evaluation of a crack bridge performance in fiber reinforced composites," in *Applied Mechanics 2011, held in Velké Bílovice, Czech Republic*, L. Náhlík, M. Zouhar, M. Ševčík, S. Seitl, and Z. Majer, Eds., Ústav fyziky materiálů AV ČR. Velké Bílovice, Czech Republic: Ústav fyziky materiálů AV ČR, 2011, pp. 243–246.