Computational modeling of statistical size effect in quasibrittle structures

D. Novák
Institute of Structural Mechanics, Brno University of Technology, Brno, Czech Republic
Z. P. Bažant
Department of Civil Engineering and Materials Science, Northwestern University, Evanston, Illinois, USA
M. Vořechovský
Institute of Structural Mechanics, Brno University of Technology, Brno, Czech Republic

Keywords: fracture mechanics, statistical size effect, theory of extreme values, failure probability, Weibull theory, random fields, stochastic finite element method, nonlinear fracture mechanics, cohesive crack

ABSTRACT: The paper begins by discussing some fundamental features of the statistics of extremes important for the computational modeling of the statistical size effect, whose asymptotic behavior is not correctly reproduced by the existing stochastic finite element methods. A simple strategy for capturing of statistical size effect using stochastic finite element methods in the sense of extreme value statistics is suggested. Such probabilistic treatment of complex fracture mechanics problems using the combination of feasible type of Monte Carlo simulation and nonlinear fracture mechanics computational modeling are presented using numerical example of crack initiation problem - size effect due to bending span of four-point bending tests.

1 INTRODUCTION

Large concrete structures usually fracture under a lower nominal stress than geometrically similar small structures (the nominal stress being defined as the load divided by the characteristic cross section area). This phenomenon, called the size effect, has in general two physical sources – deterministic and statistical. The deterministic source consists of the stress redistribution and the associated energy release described by nonlinear fracture mechanics (in finite element setting, the crack band model or cohesive crack model). The deterministic size effect represents a transition from ductile failure with no size effect, asymptotically approached for very small structures, to brittle failure with the strongest possible size effect, asymptotically approached for very large structures (Bažant & Planas 1998).

The classical explanation of size effect used to be purely statistical – simply the fact that the minimum random local strength of the material encountered in a structure decreases with an increasing volume of the structure. This idea was qualitatively proposed already in the middle of the 17th century by Mariotte. Although what became known as the Weibull distribution was in mathematics discovered already in 1928 by Fisher & Tippett (1928) (in connection with Tippett's studies of the length effect on the strength of long fibers), the need for this extreme value distribution in describing fatigue fracture of metals and the size effect in structural engineering was first developed, independently of Fisher & Tippett's cardinal contribution, by Weibull (1939). His pioneering work was subsequently refined by many other researchers, mainly mathematicians; e.g. Epstein (1948) and Saibel (1969).

The classical (but erroneous) view that any observed size effect should be described by extreme value statistics prevailed in structural engineering until about 1990. However, beginning with the studies at Northwestern University initiated in the mid 1970s, it gradually emerged that there exists a purely deterministic size effect, caused by energy release associated with stress redistribution prior to failure, and that this energetic size effect usually dominates in the so-called quasibrittle structures (i.e., structures in which fracture propagation is preceded by a relatively large fracture process zone which, in contrast to brittle-ductile fracture of metals, exhibits almost no plastic deformations but undergoes progressive softening due to microcracking). Beginning with the 1990s, many studies focused on the deterministic size effect; see the reviews in Bažant (1986), Bažant & Chen (1997), Bažant & Planas (1998), Bažant (1999a). The recent development of nonlocal Weibull theory by Bažant & Novák (2000ab) in connection with statistical studies of the modulus of rupture (or flexural strength) of plain concrete beams showed the that, for large quasibrittle structures failing at crack initiation, the deterministic energetic size effect needs to be combined with the Weibull probabilistic size effect. In this connection, some fundamental questions arose regarding the applicability of various statistical approaches to the statistical size effect. As shown by Bažant & Novák (2000a) and Bažant (2002), the existing stochastic finite element method (SFEM) does not have the correct large size asymptotic behavior and fails to capture the statistical size effect on nominal strength.

The decisive parameter in SFEM is the correlation length which governs spatial correlation over the structure. The correlation length modifies the size effect curve in the region where this parameter is smaller than the element size. There is a clear relationship – the larger the correlation length, the stronger is the spatial correlation of strength along the structure and, consequently, the weaker is the decrease (due to local strength randomness) of the nominal strength with increasing structure size. Computational problems, however, develop in trying to simulate the extreme value asymptotic size effect using the random field approach. Approximately, the requirement is that the ratio of the correlation length to the element size should not drop bellow one. This poses a major obstacle to using SFEM for describing the size effect, especially for large structure sizes.

Some advances in this problem were achieved by several authors, e.g. Gutiérrez & de Borst (2001) who, however, confined their studies to the size range of real structures. The ratio of the correlation length to the element size implies, unfortunately, a severe limitation. To actually compute the extreme value asymptote using the random field approach, the number of discretization points (e.g. nodes in a finite element mesh) would have to increase proportionally to the structure size, which is in practice impossible since an extremely large structure size would have to be considered to approach the asymptotic behavior closely. To make computations feasible, it is necessary to devise a way to increase the element size in proportion to the structure size, keeping the number of elements constant. Therefore, the aims of this paper are:

1. To introduce the problem by summarizing the vital features of the statistics of extremes established by mathematical statisticians in a form meaningful to engineers, putting emphasis on the philosophy of derivation of the probability distribution of extreme values in a set of independent stochastic variables having an arbitrary elemental probability distribution.
2. To draw the consequence for capturing the statistical size effect with the help of SFEM.
3. To propose a method for computer simulation of the statistical size effect based directly on the basic concept of extreme value statistics in combination with nonlinear fracture mechanics, and verify it by an example.

2 WEAKEST LINK CONCEPT AND THEORY OF EXTREME VALUES

The weakest link concept for the strength of a chain-like structure with N elements is equivalent to the distribution of the smallest values in samples of size N. If one element, the weakest element, fails, the whole structure fails, i.e., the failure is governed solely by the element of the smallest strength. To clarify the problem, it will be useful to recall some basic formulae of the statistical theory of extremely small values. The strength distribution of an element of a chain-like (or statically determinate) structure, i.e., the distribution of the failure probability of an element as a function of the applied stress σ, may be characterized by continuous probability density function $p_1(\sigma)$ with the associated cumulative distribution function $P_1(\sigma)$ (in statistical literature called the elemental, underlying, basic or primary distribution). Then the cumulative distribution of the failure probability of a structure of N elements (or the distribution of the smallest strength value in samples of size N) is given by

$$P_N(\sigma) = \int_{-\infty}^{\sigma} p_1(\sigma) d\sigma = 1 - (1 - P_1(\sigma))^N \qquad (1)$$

and the failure probability density is

$$p_N(\sigma) = np_1(\sigma)(1 - P_1(\sigma))^{(N-1)} \qquad (2)$$

These basic equations provide an overall representation of the failure distribution $p_N(\sigma)$ (or $P_N(\sigma)$) corresponding to a given elemental distribution $p_1(\sigma)$. Different elemental distributions can give different failure distributions $P_N(\sigma)$, however, it is remarkable that the asymptotic forms $P_N(\sigma)$ can be only three. Before discussing this fact, let us illustrate the influence of the type of elemental distribution on the failure distribution graphically.

Figure 1 shows the plots of the failure probability density functions $p_N(\sigma)$ and the cumulative distribution functions $P_N(\sigma)$ calculated for $N = 1, 10, 100$ and 1000 according to (1)–(2) for the various elemental distributions, in particular the (a) normal, (b) Weibull and (c) rectangular distributions (the last one is included merely for comparison purposes). All the elemental distributions are chosen to have the same mean value, 1, and the same standard deviation, 0.2.

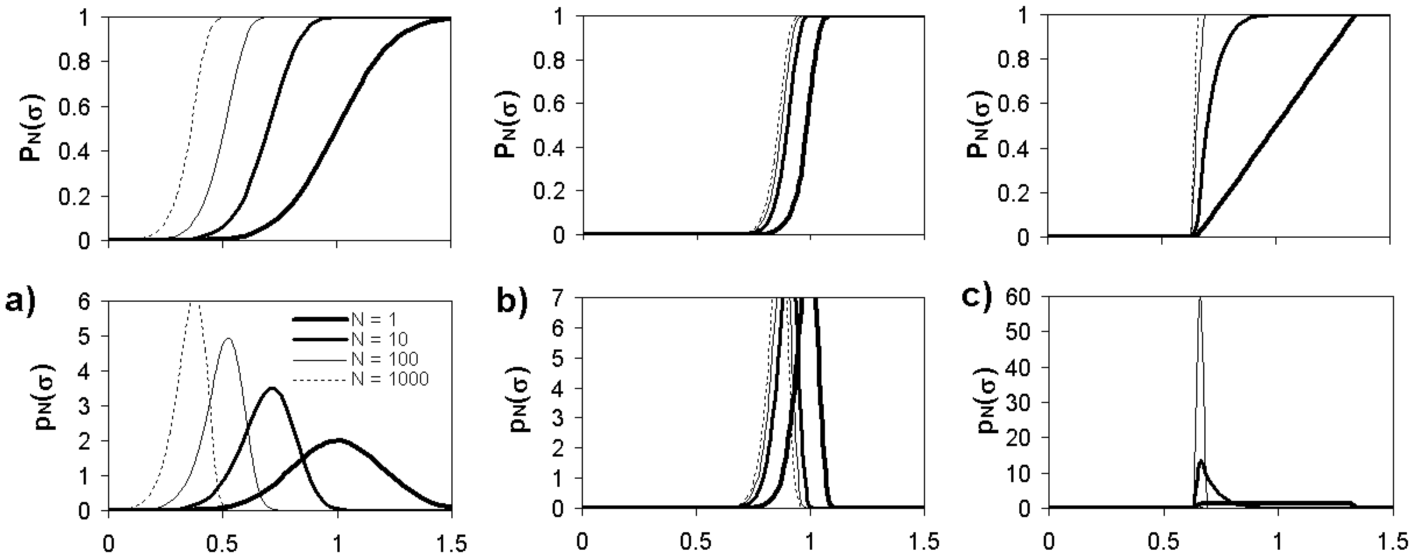

Figure 1: Failure probability density and cumulative distribution function for different elemental distributions: a) normal; b) Weibull; c) rectangular.

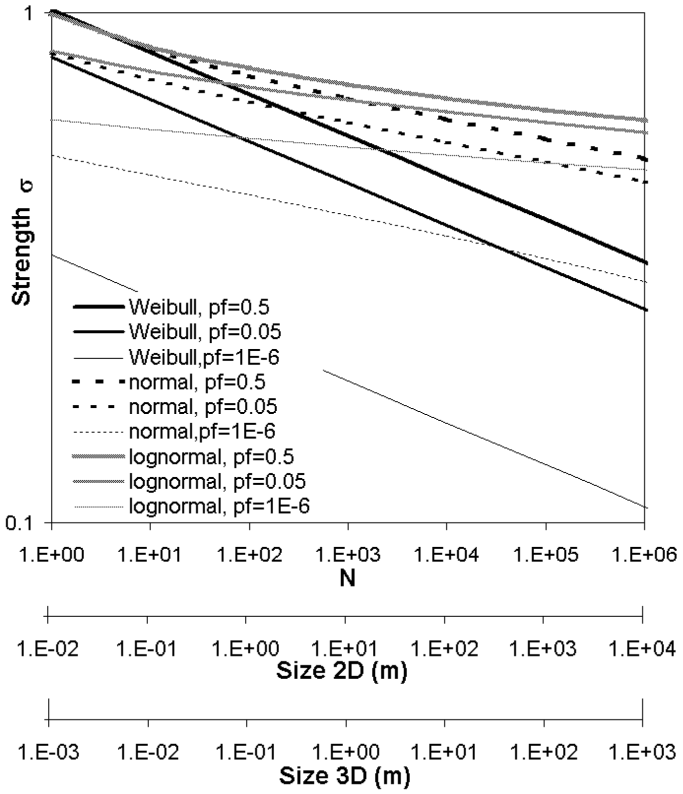

Figure 2: Dependence of strength on N for elemental distributions.

A general trend may be noticed: Both the mean value and the variance decrease with an increasing sample size (i.e., number N of elements). Cases (a) and (b) are very similar in these overall plots, having a bell-shaped form. But, as discussed later, for large N, the differences are becoming very significant especially for very small probabilities normally required in design. When the elemental distribution is rectangular (case c), the extreme value is seen to converge very quickly to the threshold of the rectangular distribution. This distribution exhibits no size effect, which makes it unacceptable (aside from physical reasons) for strength modeling. But the elemental normal and lognormal distributions give also a physically unacceptable distribution of structural strength, since for small enough probability they give a negative strength value. Thus Figure 1 provides a qualitative insight into the statistics of extremes.

Differences in structural strength for various elemental distribution are particularly pronounced for large N and small probabilities (i.e., in the tail). This phenomenon is illustrated in Figure 2, in which the basic equation (1) is used in the inverse: For a chosen failure probability $P_N(\sigma)$, the strength σ is solved. Naturally, even for the elemental distributions, the main differences lie in their tails (case $N = 1$). But as N increases, the differences in strength get larger and larger, not only for the tails but also for the medians. The dependence of strength on N is plotted in Figure 2 for selected failure probabilities $P_N = 0.5, 0.05, 10^{-6}$. Three elemental distributions, normal, Weibull and lognormal, with the same mean, 1, and the same standard deviation, 0.1, are considered, and enormous differences among them are found. For the elemental normal distribution, the fact that the size effect on the mean is stronger than on the tail is unrealistic. A more realistic, and much stronger, size effect is observed for the Weibull elemental distribution. For the elemental mean 1 and standard deviation 0.1, the statistical parameters for the Weibull (two-parametric) distribution are: m = 12.15 (Weibull modulus) and σ_0 = 1.043 (scale parameter). In the double logarithmic plot of Figure 2, the Weibull size effect is, for any specified failure probability, a straight line of slope $-1/m$.

To show the differences among structures that are scaled in one-, two- and three-dimensions (1D, 2D, 3D), Figure 2 includes three horizontal scales. For the validity of (1) and (2) in muti-dimenensional situations, it is required that the whole structure fails when a single element fails. This is a property of a chain as well as all statically determinate structural systems, and is also a good approximation for fracture of unnotched structures of positive geometry (e.g., unreinforced concrete beams in flexure). In that case, N represents the ratio of the structure volume to the characteristic volume V_c of the micro-heterogeneous material. V_c is here understood as the volume having the size of the autocorrelation length of the random field of the local material strength, in which case the strength limits of various characteristic volumes can be considered as statistically independent (uncorrelated) random variables, a basic hypothesis in the statistical theory of extremes (note that V_c is in general different (and larger) than the representative volume V_r of the material, which is the smallest volume for which the continuum concepts of stress and strength make sense, or a volume for which the mean strength is unaffected by randomness of microstructure as this volume is shifted through the material). With respect to the situation in concrete structures, V_r may be considered to be approximately 0.01 m^2 (for 2D) and 0.001 m^3 (for 3D).

The foregoing illustrations bring to light a salient point (which will be discussed in detail later)—namely, the selection of the elemental probability distribution is of fundamental importance for the statistical size effect, and must therefore be realistic.

3 IMPLICATIONS FOR FINITE ELEMENT METHOD

Since the failure probabilities acceptable for design are of the order of 10^{-7}, at least 1 billion material tests of identical specimens would be needed to verify the elemental statistical distribution purely experimentally. This is obviously impossible. However, a verification is made possible by scaling up the structure to a very large size, a size that would comprise 1000^3 characteristic volumes. Thus a verification of the strength distribution of such a structure is equivalent to conducting 1 billion material tests, provided that the structure is of a type for which the failure of one element causes the whole structure to fail. The strength distribution of such a structure is known, based on a mathematical argument. Therefore, one needs to consider the large size asymptotic behavior and verify that it conforms to this distribution.

The asymptotic behavior rests on the so-called stability postulate of extreme value statistics, generally accepted beginning with Fréchet (1927). In this postulate, the extreme value of a set of $\nu = Nn$ identical independent random variables x (the strengths) is regarded as the extreme of the set of n extremes of the subsets of N variables. When both $n \to \infty$ and $N \to \infty$, it is perfectly reasonable to postulate that the distribution of the extreme of set Nn must be similar to the distribution of the extreme of each subset N (i.e., related to it by a linear transformation). In other words, the asymptotic form of the distribution must be stable. From this property it can be shown that the survival probability f_N of a structural system with a very large size N as a function of applied strength σ must asymptotically satisfy the functional equation

$$[f(\sigma)]^N = f(a_N\sigma + b_N) \tag{3}$$

where a_N and b_N are functions of size N. In the most important paper of extreme value statistics motivated by the strength of textile fibers, Fisher & Tippett (1928) showed that this recursive functional relation for function $f(\sigma)$ can be satisfied by three and only three distributions. One of them had already been found by Fréchet (1927) and the other two have later become known as the Gumbel and Weibull distributions (curiously, not the Fischer and Tippett distributions). The first two distributions have no threshold and admit negative values of the argument, and so are unsuitable for strength. Hence, the Weibull distribution is the only realistic distribution for structural strength.

Consequently, the only way to ensure the correctness of SFEM for failure analysis is to make it match the large size asymptotic behavior, in particular, the Weibull power law size effect, typical of structures failing at crack initiation. But how to overcome the obstacle of a forbiddingly large number of random variables associated with all the finite elements?

The basic idea proposed here is to exploit directly the fundamental stability postulate from which Fisher & Tippett derived the asymptotic forms of the extreme value distributions. In regard to SFEM, this postulate may be literally implemented as follows: Instead of subdividing a very large structure into the impracticably large number ν of finite elements having the fixed size of the characteristic volume, we must use a mesh with only n macroelements (finite elements) associated with n random strength variables, keeping n fixed and increasing the macroelement size with the structure size, while the subdivision N of each macroelement is defined as the ratio of its volume to the characteristic volume of the material. Then each of these n subsets of N variables is simulated statistically, and for each subset the extreme is selected to be the representative statistical property of the finite element (macroelement). These n extremes of the subsets of N variables are then used in FEM analysis of the whole structure. This procedure ensures that the extreme value statistics is correctly approached, with one crucial advantage—the number n of finite elements (macroelements) remains reasonable from the computational point of view. Although N increases with the structure size, the determination of the extreme from the subdivision of each macroelement does not add to the computational burden since it is carried out outside FEM analysis.

One basic hypothesis of the classical Weibull theory of structural strength is the statistical independence of the strengths of the individual characteristic volumes ${l_0}^2$ (in 2D) or ${l_0}^3$ (in 3D), where l_0 is the characteristic length. The strength of each of these volumes can be described by Weibull distribution with Weibull modulus m and scale parameter σ_0 (the threshold being taken as zero, as usual). Each of the aforementioned macroelements, whose characteristic size is L_0 and characteristic volume ${L_0}^2$ or ${L_0}^3$, may be imagined of being discretized into N characteristic volumes ${l_0}^2$ or ${l_0}^3$, i.e. $N = {L_0}^2/{l_0}^2$ or ${L_0}^3/{l_0}^3$. This consideration provides, according to (1) or (2), the statistical properties of the macroelement. Since we are interested only in very small tail probabilities, we may substitute in these equations the tail approximation of the elemental (generic) Weibull distribution with a certain modulus and scale parameter. The tail approximation is the power function σ^m (times a constant), and its substitution leads for the strength of the macroelement again to Weibull distribution but with a different modulus and scale parameter, and thus with a different mean and variance, which are expressed as follows:

$$\mu = \sigma_0 (N)^{-1/m} \Gamma(1 + 1/m), \tag{4}$$

$$\sigma^2 = \mu^2 \left(\frac{\Gamma(1 + 2/m)}{\Gamma^2(1 + 1/m)} - 1 \right) \tag{5}$$

4 NUMERICAL EXAMPLE: SIZE EFFECT OF SPAN IN FOUR-POINT BEND BEAM TESTS

4.1 *Experiment and attempt at deterministic simulation*

Abundant experimental evidence on the statistical size effect on plain concrete beams has been accumulated by now in the literature. Recently, Koide et al. (1998, 2000) tested 279 plain concrete beams under four-point bending, aimed at determining the influence of the beam length L on the flexural strength of beams. These excellent data permit a comparison of the cumulative probability distribution function (CPDF) of the maximum bending moment M_{max} at failure (Bažant & Novák 2000b, Novák et al. 2001). Beams of three different bending spans, 200, 400 and 600 mm (series C of Koide et al.) are shown in Figure 3, along with the cracks obtained by deterministic finite element calculations, Figure 4 (with the code ATENA, Červenka & Pukl 2002). The cross-sections of all the beams were kept constant (0.1m $\times$0.1m). The experimental data show that M_{max} decreases as the span increases. To explain this size effect of the span, shown in Figure 7, Koide et al. provided a Weibull theory based approach.

Unfortunately, only the compression strength of the concrete used is known, whereas the direct tensile strength and fracture energy have not been tested. The experimental data depicted in Figure 7 represent the mean values for each size. The double logarithmic plot of M_{max} versus the span forms a straight line with a slope $D^{-n/m}$, where n is the spatial dimension and m is the Weibull modulus. The problem is properly analyzed as one-dimensional, and then the overall slope of the experimental data in the figure is matched best using $m = 8$ (which is an unusually low value for concrete, indicating a relatively high scatter).

Deterministic simulation with nonlinear fracture mechanics software ATENA yields results consistent with a flat size effect curve, i.e., absence of size effect. This is not surprising. According to fracture mechanics, there is almost no deterministic size effect in flexure of unreinforced beams when the beam depth is not varied because the energy release function is almost independent of the beam span. This is useful for our focus on the statistical size effect. It allows a purely statistical analysis of the test data in Figure 7, reflecting the fact that, the longer the beam, the higher is the probability of encountering in it a material element of a given low strength.

In finite element simulations, the beams were loaded by force increments in order to avoid a nonsymmetric bending moment distribution when the crack pattern (Fig. 4) becomes nonsymmetric, due to material randomness. The load-deflection curves, including the peak and postpeak, were calculated under load control using the arc length method.

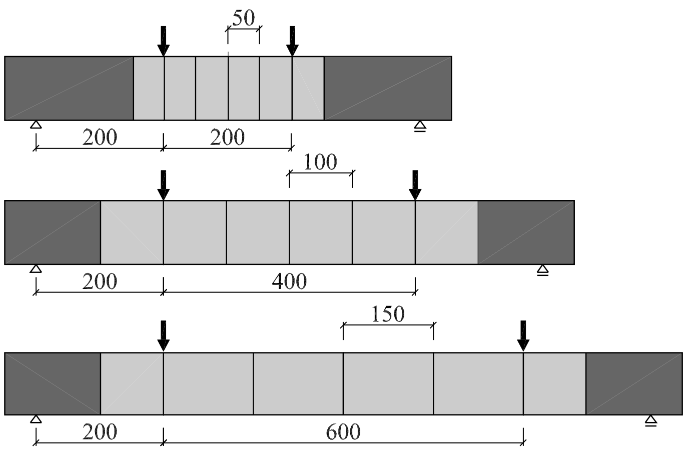

Figure 3: Koide's beams of bending span 200, 400 and 600 mm, series C.

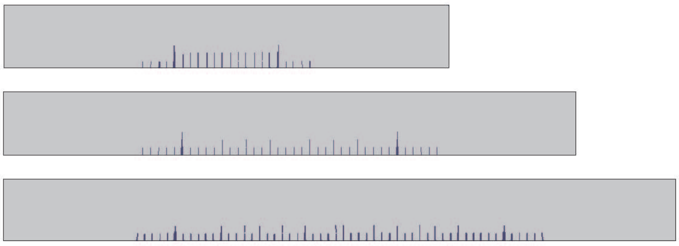

Figure 4: Deterministic cracks for sizes 20, 40 and 60.

4.2 *Statistical size effect*

The probabilistic version of nonlinear fracture mechanics software ATENA (Pukl et al. 2003) was utilized to simulate the tests of Koide et al. by finite elements, in accordance with the theory of extreme values. This was made possible by integrating ATENA with the probabilistic software FREET (Novák et al. 2002, 2003).

In this simulation, the finite element mesh is defined by using only 6 stochastic macroelements placed in the central region of test beams in which fracture initiates randomly; see Figure 5. The chosen macroelements have the form of strips. The strips suffice for simulating the Weibull size effect. We imagine N elements per macroelement of width L_0, while the finite element meshes for all the sizes are identical (except for a horizontal stretch).

The characteristic length is considered to be approximately 3-times the maximum aggregate size, i.e., about 50 mm. The Weibull modulus is taken as m=8, and the scale parameter is 1.0. The statistical parameters of the strength of the macroelements, imagined to consist of $N = L_0/l_0$ material elements each, are calculated from (4). For the three sizes (spans) considered here, $L_0 = 50, 100, 150$ mm and $N = 1, 2, 3$.

In the present approach, a stochastic computational model with n=6 random tensile strength variables is defined for each beam size (span); 16 random simulations of these 6 statistically independent variables, based on the method of Latin hypercube sampling, are performed using FREET and ATENA softwares (Novák et al. 2003, Vořechovský & Novák 2003, Pukl et al. 2003). The statistical characteristics of the ultimate force can then be evaluated. The mean values of nominal strength obtained from a statistical set of maximum forces are determined first. Figure 5 shows the random cracking pattern at failure, obtained for four realizations of three progressively improved alternatives of solution.

To illustrate the random failures, the corresponding random load-deflection curves are shown in Figure 6. The three alternatives, for which the results are presented in Figure 7, are as follows:

Alternative I: The first alternative is a pure Weibull type approach in which only the random scatter of tensile strength is considered, the generic mean value of tensile strength being fixed as 3.7 MPa. For the three sizes (spans) considered here then, according to formulas (4) and (5) the means of tensile strengths are $\mu = 3.484, 3.195$ and 3.037 MPa, coefficient of variation COV = 0.148 (driven by the Weibull modulus m only).

The resulting size effect curve obtained by probabilistic simulation is found to have a smaller slope than the experimental data trend, in spite of the fact that an unusually low Weibull modulus ($m = 8$) is used. This can be explained easily. The Weibull theory strictly applies only when the failure occurs at crack initiation, before any (macroscopically) significant stress redistribution with energy release. However, the material, concrete, is relatively coarse, the test beams not being large enough compared to the aggregate size, and so a nonnegligible fracture process zone must form before a macroscopic crack can form and propagate, dissipating the required fracture energy G_f per unit crack surface. Therefore, the beam, analyzed by nonlinear fracture mechanics (the crack band model, approximating the cohesive crack model) does not fail when the first element fails (as required by the weakest link model imitating the failure of a chain). Rather, it fails only after a group of elements fails, and several groups of failing elements can develop before the beam fails; see Figure 5. The finite element simulations are able to capture this behavior thanks to the cohesive nature of softening in a crack, reflecting the energy release requirement of fracture mechanics.

Alternative II: The idea to overcome the problem and match the size effect data is to take the randomness of fracture energy G_f into account. Using the generic mean of fracture energy, $G_f = 93$ N/m, for the three spans, according to formulas (4) and (5) the means of fracture energy are $G_f = 87.6, 80.3$ and 76.3 N/m, COV 0.148. The generic mean of tensile strength is again $\mu = 3.7$ MPa. But we cannot ignore the statistical correlation of G_f to tensile strength. For lack of available data, we simply assume a very strong correlation, characterized by correlation coefficient 0.9. Such a correlation tends to cause an earlier onset of (macroscopic) crack propagation, compared to Alternative I. The result is shown in Figure 7 as Alternative II. The resulting slope of the simulated size effect curve is now close to the slope of experimental data. However, the whole curve is shifted down, i.e., all the beams are weaker than they should be. It can be seen that the strong correlation between tensile strength and fracture energy causes the macroelements with a lower tensile strength to be more brittle. The failure, therefore, localizes into these macroelements (Fig. 5).

Alternative III: In seeking a remedy, we must re-

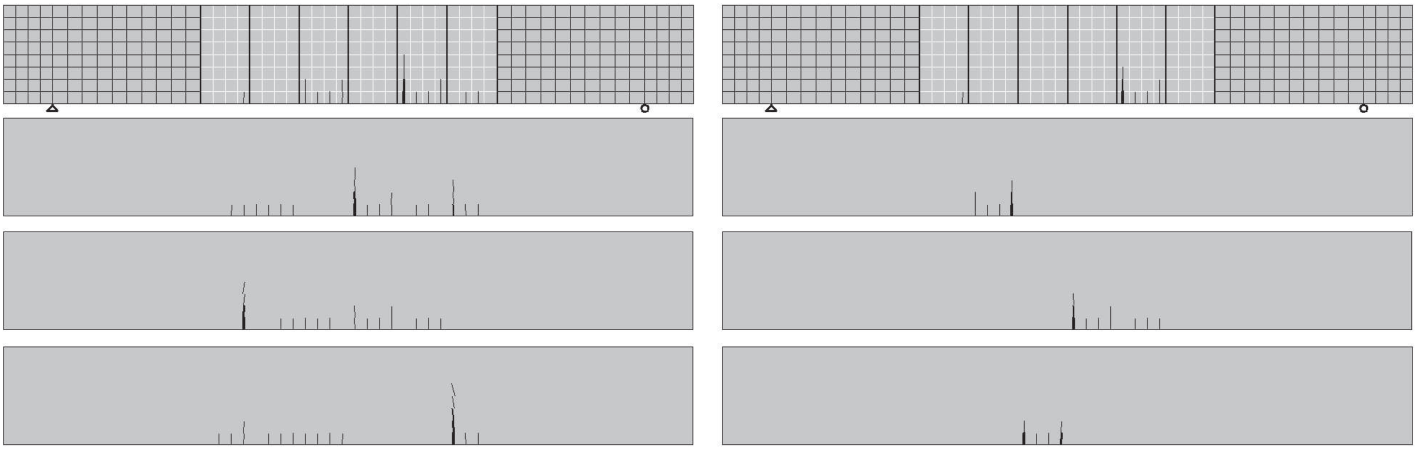

Figure 5: Macroelements and examples of random crack initiation for the first size; left: random tensile strength only, right: random and correlated tensile strength and fracture energy.

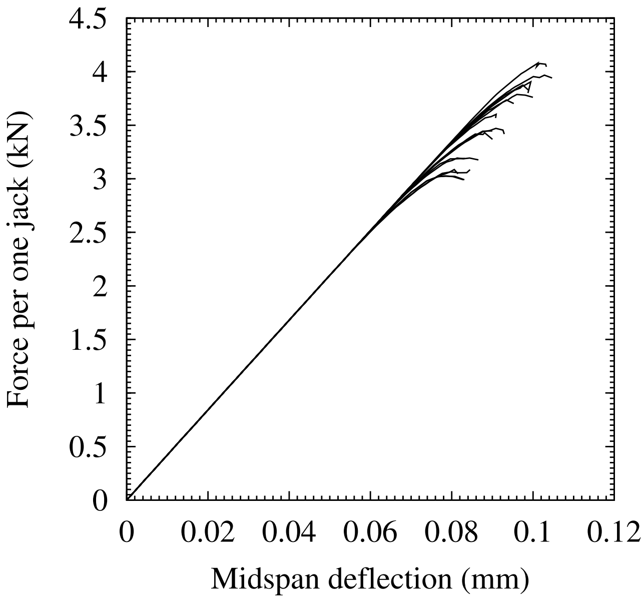

Figure 6: Random load deflection curves.

alize that Koide et al. have not measured the tensile strength nor the fracture energy, and our foregoing estimate may have been too low. So a heuristic approach is the only option. While keeping Aternative II, we are free to shift the size effect curve up by increasing the generic mean value of tensile strength and the fracture energy value. We increase them to 4.1 MPa and 102 N/m, respectively, and this adjustment is found to furnish satisfactory results; see Figure 7. Although the size effect of Alternative III in the double logarithmic plot is not as straight as the trend of data, the differences from the data are negligible. These small differences may have been easily caused, for instance, by insufficient size of the calculated data set, or by weaker numerical stability near the peak load, making a precise detection of the peak (under load control) less accurate. Finally, it may be emphasized that the result of Alternative III is in excellent agreement with the previous analysis of these data according to the nonlocal Weibull theory (Bažant & Novák 2000b).

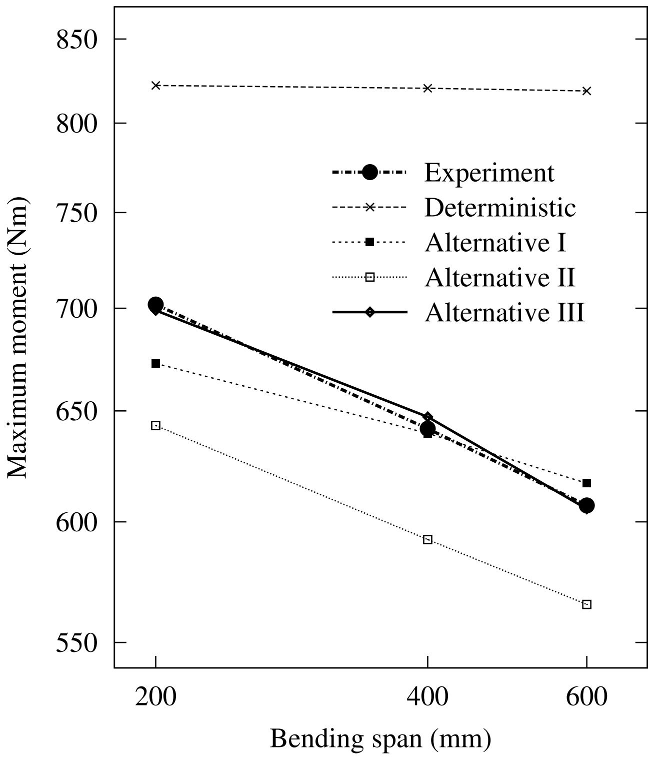

Figure 7: Comparison of means of Koide's data and ATENA deterministic and statistical simulations.

5 CONCLUSION

The paper tackles a problematic feature of stochastic finite element method: How to capture the statistical size effect for structures of very large sizes. A simple and effective strategy for capturing the statistical size effect using stochastic finite element methods is developed. The idea is to emulate the recursive stability property from which the extreme value distribution, the Weibull distribution, is derived. Using the combination of a well feasible type of Monte Carlo simulation and of computational modeling of nonlinear fracture mechanics, a probabilistic treatment of complex fracture mechanics problems is rendered possible. The feasibility of the approach is documented by simulating the size effect in plain concrete beams under four-point bending, for which extensive statistical test data have recently been reported by Koide et al.

ACKNOWLEDGMENTS

The work of the first and third authors was supported by the Grant Agency of the Czech Republic under the grant number 103/02/1030 and under project reg. No. CEZ: J22/98: 261100007; the work of the second author was partially supported by the U.S. National Science Foundation under grant CMS-9713944 to Northwestern University.

REFERENCES

Bažant, Z. P. 2002. Probabilistic Modeling of Quasibrittle Fracture and Size Effect. *ICoSSaR 2001 (Proc., 8^{th} Int. Conf. on Structural Safety and Reliability, held in California, USA*, R.B. Corotis et al, eds., Swets & Zeitinger Publishing.

Bažant, Z.P. 1986. Mechanics of distributed cracking. *Appl. Mech. Reviews ASME*, 39, 675–705.

Bažant, Z.P., & Chen, E.-P. 1997. Scaling of structural failure. *Applied Mechanics Reviews ASME* 50 (10), 593–627; transl. in *Advances in Mechanics* (China) 29 (3), 383–433.

Bažant, Z.P. 1999a. Size effect. *International Journal of Solids and Structures* 37 (200), 69-80; special issue of invited review articles on *Solid Mechanics* edited by G.J. Dvorak for U.S. Nat. Comm. on Theor. and Appl. Mech., publ. as a book by Elsevier Science, Ltd.

Bažant, Z.P. 1999b. Size effect on structural strength: a review. *Archives of Applied Mechanics* (Ingenieur-Archiv, Springer Verlag) 69, 703–725 (75th Anniversary Issue). Reprinted with updates in *Handbook of Materials Behavior*, J. Lemaitre, ed., Academic Press, San Diego 2001, Vol. 1, 30–68.

Bažant, Z. P. & Novák, D. 2000a. Probabilistic Nonlocal Theory for Quasibrittle Fracture Initiation and Size Effect. I: Theory, *Journal of Engineering Mechanics*, Vol. 126, No. 2, 166-174.

Bažant, Z. P. & Novák, D. 2000b. Probabilistic Nonlocal Theory for Quasibrittle Fracture Initiation and Size Effect. II: Application, *Journal of Engineering Mechanics*, Vol. 126, No. 2, 175-185.

Bažant, Z. P. & Novák, D. 2000c. Energetic-Statistical Size Effect in Quasibrittle Failure at Crack Initiation, *ACI Materials Journal*, May-June 2000, Vol. 97, No. 3, 381-392.

Bažant, Z. P. & Planas, J. 1998. *Fracture and Size Effect in Concrete and Other Quasibrittle Materials.* CRC Press, Boca Raton, Florida, and London (chapter 12).

Červenka, V. & Pukl, R. 2002. *ATENA - Computer Program for Nonlinear Finite Element Analysis of Reinforced Concrete Structures*. Program documentation. Prague, Czech Republic: Červenka Consulting.

Epstein, B. 1948. Statistical Aspects of Fracture Problems. *Journal of Applied Physics*, Vol. 19, 140-147.

Fischer, R. A. & Tippett L. H. C., 1928. Limiting Forms of the Frequency Distribution of the Largest and Smallest Member of a Sample, *Proc., Cambridge Philosophical Society*, Vol. 24, 180-190.

Fréchet, M. 1927. Sur la loi de probabilité de l' écart maximum. *Ann. Soc. Polon. Math.* 6, 93.

Gutiérrez, M.A. & de Borst, R. 2001. Deterministic and probabilistic length-scales and their role in size-effect phenomena. *ICOSSAR 2001 (Proc., 8^{th} Int. Conf. on Structural Safety and Reliability, held in California, USA), R.B. Corotis et al, eds., Swets & Zeitinger*, 1-8.

Koide, H., Akita, H. & Tomon, M. 1998. Size Effect on Flexural Resistance due to Bending Span of Concrete Beams, *Fracture Mechanics of Concrete Structures* (Proc., 3rd Int. Conf., FraMCoS-3, held in Gifu, Japan), H. Mihashi and K. Rokugo, eds., Aedificatio Publishers, Freiburg, Germany, 2121-2130.

Koide, H., Akita, H., & Tomon, M. 2000. Probability Model of Flexural Resistance on Different Lengths of Concrete Beams. *Application of Statistics and Probability* (Proc., 8th Int. Conf., ICASP-8, held in Sydney, Australia 1999), R. E. Melchers and M. G. Stewart, eds., Balkema, Rotterdam, Vol. 2, 1053-1057.

Novák, D. et al. 2002. *FREET - Feasible Reliability Engineering Efficient Tool*. Program documentation, Brno University of Technology, Faculty of Civil Engineering / Červenka Consulting, Prague, Czech Republic.

Novák, D., Rusina, R. & Vořechovský, M. 2003. Small-sample probabilistic assessment - FREET software . *Proc., ICASP 9*, San Francisco, USA.

Novák, D., Vořechovský, M., Pukl, R. & Červenka, V. 2001. Statistical nonlinear fracture analysis: Size effect of concrete beams. *Int. Conf. On Fracture Mechanics of Concrete Structures FraMCos 4*, Cachan, France.

Pukl R., Strauss A., Novák D., Červenka V. & Bergmeister K. 2003. Probabilistic-based assessment of concrete structures using non-linear fracture mechanics. *Proceedings 2003, ICASP 9, San Francisco, USA.*

Saibel, E. 1969. Size Effect in Fracture, *Proc. Int. Conf. on Structure, Solid Mechanics and Engineering Design in Civil Engineering Materials*, Southampton, part 1, 125-130.

Vořechovský, M. & Novák, D. 2003. Statistical correlation in stratified sampling. *Proc., ICASP 9*, San Francisco, USA.

Weibull W. 1939. The Phenomenon of Rupture in Solids. *Proc., Royal Swedish Institute of Engineering Research* (Ingenioersvetenskaps Akad. Handl.), Stockholm, Vol. 153, 1-55.